RATIONAL EMBEDDINGS OF CONTINUOUS AUTOMATIC GROUPS (PRELIMINARY VERSION)

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ABSTRACT. We introduce the notion of **continuous normal form** and use it to show that large classes of groups, including cubulated groups and finite type Artin groups, embed into the rational group.

CONTENTS

1. INTRODUCTION

A **compactification** of a discrete group G is an embedding of G as a dense subset of some compact Hausdorff space $G \cup \partial G$, together with a continuous action of G on $G \cup \partial G$ that extends the left action of G on itself. In this case, the set ∂G is called a **boundary** of G. Examples of such boundaries include the ends of a group, the Gromov boundary of a δ-hyperbolic group, and the horofunction boundary of a discrete group.

In this paper we introduce a new class of boundaries for discrete groups, which we call **normal form boundaries**. Such a boundary makes sense for any finitely generated group that possesses a certain kind of normal form, which we call a **continuous normal form**. We are particularly interested in the case where G is an automatic group whose automatic structure is a continuous normal form, which we refer to as a **continuous automatic group**. In this case, the resulting normal form boundary is naturally a compact subset of a sofic shift.

Normal form boundaries are always compact, Hausdorff, and totally disconnected. Like horofunction boundaries, normal form boundaries are not quasi-isometry invariants, and indeed different normal forms on the same group can give very different normal form boundaries. Totally disconnected boundaries can be useful because the action of G on ∂G determines a **topological full group** $[\![G \mid \partial G]\!]$ consisting of all piecewise-G homeomorphisms of ∂G. Such a group usually contains Thompson's group V, and can be thought of as a "Thompson-like" group into which G embeds.

For example, in [BBM21], the first, second, and fourth authors described symbolic dynamics for the action of a δ-hyperbolic group G on its horofunction boundary $\partial_h G$. In particular, we described a symbolic coding of $\partial_h G$ such that elements of G act by rational homeomorphisms; it follows that G embeds into the asynchronous rational group defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ [GNS00]. Together with Matthew Zaremsky in [BBMZ], we used this to prove that every δ-hyperbolic group G embeds into a finitely presented simple group. In particular, G embeds into the topological full group $[\![G \mid \partial_h G]\!]$, which in turn embeds into a finitely presented, simple twisted Brin–Thompson group. This establishes the generic case of the Boone–Higman conjecture, which asserts that every finitely presented group with solvable word problem embeds into a finitely presented simple group.

Here we describe the symbolic dynamics for the action of a continuous (asynchronous) automatic group on its normal form boundary. For such a group G, we prove that G acts on this boundary by rational homeomorphisms. Assuming the action is faithful, this proves that G embeds into the asynchronous rational group. Groups to which this applies include Artin groups of finite type, Baumslag–Solitar groups, and virtually torsion-free cubulated groups, i.e. groups that act geometrically on a CAT(0) cube complex.

All of this parallels the first step in resolving the Boone–Higman conjecture for δ-hyperbolic groups [BBM21], though we do not prove any Boone–Higman results here. In particular, we do not prove that any of the associated topological full groups $[\![G \mid \partial_h G]\!]$ are finitely presented. The symbolic dynamics here is more complicated than for δ-hyperpbolic groups—in particular the actions are not contracting—so new techniques will be needed to prove finite presentability.

1.1. **Outline of the theory.** Here we outline the theory we have developed, including all of the main definitions and theorems.

Normal form boundaries. Let G be a group with finite, symmetric generating set X. Let X^* be the set of all finite words over X, and let X^ω be the Cantor space of all infinite words over X. The union $X^{\leq\omega} = X^* \cup X^\omega$ is a totally disconnected, compact Hausdorff space, where a sequence $\{\alpha_n\}$ of finite words converges to an

infinite word β if and only if the length of the greatest common prefix of α_n and β goes to ∞ as $n \to \infty$.

A **language of normal forms** for G is a subset $L \subseteq X^*$ consisting of one word to represent each element of G. That is, π maps L bijectively to G, where $\pi \colon X^* \to G$ is the canonical surjection. Such a language has a **boundary** ∂L, consisting of all accumulation points of L in X^ω. We say that L is a **continuous normal form** if the natural left-action of G on ∂L extends to a continuous action of G on $L \cup \partial L$, in which case we refer to ∂L as a **normal form boundary** for G.

Example 1.1 (A free abelian group). Let $G = \langle x, y \mid xy = yx \rangle$ be a free abelian group of rank two, and consider the following normal forms on G:

(1) The normal form L_1 of all words $x^m y^n$ for $m, n \in \mathbb{Z}$.
(2) The normal form L_2 of all words of the form $(ab)^m a^n$ or $(ab)^m b^n$, for $a \in \{x, x^{-1}\}$, $b \in \{y, y^{-1}\}$, and $m, n \geq 0$.
(3) The normal form L_3 of all words of the form $x^m y^n$ ($m \geq 0$ and $n \in \mathbb{Z}$) or $y^n x^{-m}$ ($m > 0$ and $n \in \mathbb{Z}$).

The first language L_1 is a continuous normal form. Its boundary consists of the two points $x^{\pm\infty}$ as all words $x^m y^{\pm\infty}$ for $m \in \mathbb{Z}$. The action of G on ∂L_1 is not faithful, with y acting trivially. The second language L_2 is also a continuous normal form, with boundary consisting of all words $(ab)^\infty$, $(ab)^m a^\infty$ and $(ab)^m b^\infty$ for $a \in \{x, x^{-1}\}$, $b \in \{y, y^{-1}\}$, and $m \in \mathbb{Z}$. (This coincides with the horofunction boundary of $\mathbb{Z}^2$.) Note that the action of G on ∂L_2 is faithful. Finally, the third language L_3 is not a continuous normal form for G. Specifically, ∂L_3 consists of the words x^∞, $x^m y^\infty$, $x^m y^{-\infty}$, and $y^n x^{-\infty}$ ($m \geq 0$ and $n \in \mathbb{Z}$), and there is no natural action of G on ∂L_3.

Remark 1.2. Every infinite, finitely generated group G has at least one continuous normal form with respect to any generating set X. To prove this, fix a generator x and an enumeration $\{g_n\}$ of the elements of G. Then we can choose a normal form L for G such that the word for each g_n starts with x^n. The resulting normal form boundary ∂L is the one-point set $\{x^\infty\}$, so the action of G on ∂L is not faithful. We do not know whether every infinite, finitely generated group has a continuous normal form L such that G acts faithfully on ∂L.

There is a simple characterization of continuity for normal forms which does not involve boundaries.

Proposition 1.3. *Let G be a group with monoid generating set X. Let $L \subseteq X^*$ be a normal form, and for $g, h \in G$ let $|g \wedge h|_L$ denote the length of the greatest common prefix of the normal forms for g and h. Then L is continuous if and only if for every $x \in X$ and $N \in \mathbb{N}$ there exists an $M \in \mathbb{N}$ so that*

$$|g \wedge h|_L \geq M \quad \Rightarrow \quad |xg \wedge xh|_L \geq N$$

for all $g, h \in G$.

See Section 2.1 for a proof.

Continuous automatic groups. If X is a finite alphabet, a language $L \subseteq X^*$ is called **regular** if it is accepted by some finite-state acceptor over X^*. Furthermore, a relation $R \subseteq X^* \times X^*$ is called **synchronous rational** if it is accepted by some finite-state synchronous acceptor over $X^* \times X^*$, and **deterministic rational** if it

accepted by some finite-state deterministic acceptor over $X^* \times X^*$. See Section 4.1 for precise definitions.

If G is a group with finite, symmetric generating set X, an **automatic structure** on G is a language $L \subseteq X^*$ satisfying the following conditions:

(1) L is a regular language, and $\pi(L) = G$, where π is the canonical surjection $X^* \to G$.

(2) For each $x \in X$, the relation

$$R_x = \{(\alpha, \beta) \in X^* \times X^* \mid \pi(\alpha) = x\,\pi(\beta)\}$$

is synchronous rational.

If such an L exists then G is called an **automatic group**. The definition of automatic groups is due to Thurston [Ree22], and the basic theory was developed by Epstein, Cannon, Holt, Levy, Paterson, and Thurston in [ECH$^+$92]. Though $\pi\colon L \to G$ is not required to be bijective, it is well-known that every automatic group G has an automatic structure L which is a normal form for G (see [ECH$^+$92, Theorem 2.5.1]).

If we instead require the weaker condition that the relations R_x be deterministic rational, then L is called an **asynchronous automatic structure** and G is **asynchronous automatic** (see [ECH$^+$92, Chapter 7]). Again, any asynchronous automatic group has an asynchronous automatic structure which is a normal form.

We will say that a group G is **continuous automatic** if there exists an automatic structure on G which is a continuous normal form. Continuous asynchronous automatic groups are defined similarly. In either case, the automatic structure L has a normal form boundary ∂L, and G acts continuously on ∂L. In Section 2.4 , we prove that the class of continuous automatic groups has the following properties.

Proposition 1.4.

(1) *Finite groups are continuous automatic, as is the infinite cyclic group $\mathbb{Z}$.*

(2) *If G is continuous automatic and H is commensurable to G, then H is continuous automatic.*

(3) *If G and H are continuous automatic then so is $G \times H$.*

(4) *If G and H are continuous automatic then so is $G * H$, and for $|G| \geq 2$ and $|H| \geq 3$ this free product acts faithfully on the corresponding normal form boundary.*

Furthermore, all of these properties hold with the words "continuous automatic" replaced by "continuous asynchronous automatic".

It follows from (1) and (3) that any finitely generated abelian group is continuous automatic, and it follows from (1) and (4) that any free group of finite rank is continuous automatic. Property (4) is particularly useful for embeddings.

Corollary 1.5. *If G is any continuous automatic group, then G embeds in some continuous automatic group H which acts faithfully on its normal form boundary. The same holds for continuous asynchronous automatic groups.*

Proof. If G is trivial we are done. Otherwise, by statements (1) and (4) above the free product $G * \mathbb{Z}$ is continuous automatic (or continuous asynchronous automatic) and acts faithfully on its normal form boundary. □

Structure of the boundary. If X is a finite alphabet, a closed set $E \subset X^\omega$ is said to be **rational** if it is accepted by some finite-state path automaton, i.e. a Büchi automaton where every state is an accept state. See Section 4 for a precise definition. In dynamical systems, such sets arise as the follower sets in sofic shifts. It is easy to prove that the boundary ∂L of any regular language $L \subseteq X^*$ is a closed rational subset of X^ω.

There are also analogs for infinite words of synchronous rational and deterministic rational relations. A closed set $R \subseteq X^\omega \times X^\omega$ is **synchronous rational** if it is accepted by a nondegenerate, finite-state synchronous path automaton, and **deterministic rational** if it is accepted by a nondegenerate, finite-state deterministic path automaton. See Section 4 for precise definitions. If $L \subseteq X^*$ is a continuous automatic structure on a group G, then the set

$$E_x = \{(\sigma, \sigma') \in X^\omega \times X^\omega \mid x \cdot \sigma = \sigma'\}$$

is synchronous rational for each $x \in X$. Similarly, if L is an asynchronous automatic structure, then each E_x is deterministic rational.

All of this turns out to be closely related to the class of rational homeomorphisms defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ [GNS00]. If X is a finite alphabet, a homeomorphism $f \colon X \to X$ is **rational** if f can be described by a finite-state asynchronous transducer. The collection of all such homeomorphisms form a group, known as the **rational group** $\mathcal{R}_X$. Grigorchuk, Nekrashevych, and Sushchanskiĭ proved that the isomorphism type of $\mathcal{R}_X$ does not depend on the size of the finite alphabet X, assuming $|X| \geq 2$.

In [BBM21], the first, second, and fourth authors together with Matthew Zaremsky generalized the notion of rational homeomorphisms to clopen subsets of shifts of finite type. Here we observe that the notion of rational homeomorphism can be generalized to arbitrary closed rational sets $E \subseteq X^\omega$. We prove the following.

Theorem 1.6. *If X is a finite alphabet and $E \subseteq X^\omega$ is a closed rational set, then the collection $\mathcal{R}_E$ of rational homeomorphisms of E forms a group under composition. Furthermore, if E has no isolated points then $\mathcal{R}_E$ is isomorphic to the rational group defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ.*

We prove this theorem using the following characterization of rational homeomorphisms.

Theorem 1.7. *Let X be a finite alphabet and let $E \subseteq X^\omega$ be a closed rational set. Then a homeomorphism $f \colon E \to E$ is rational if and only if the graph of f is a deterministic rational set in $X^\omega \times X^\omega$.*

As far as we know, this characterization is new even for homeomorphisms of the full shift X^ω. One remarkable aspect of this characterization is that it is completely symmetric between the domain and range. It has always been slightly mysterious why the inverse of a rational homeomorphism is rational, but the above characterization makes this seem very natural. Indeed, deterministic path automata can be viewed as a nice alternative to transducers for defining rational homeomorphisms, which have the advantage of not favoring the domain over the range.

This characterization immediately yields the following corollary.

Corollary 1.8. *If G is a continuous asynchronous automatic group, then G acts on the corresponding normal form boundary by rational homeomorphisms.*

Indeed, we obtain the following embedding theorem.

Corollary 1.9. *Every continuous asynchronous automatic group embeds into the rational group defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ.*

Proof. Let G be a nontrivial continuous automatic group. As discussed Corollary 1.5, G embeds into the continuous asynchronous automatic group $G * \mathbb{Z}$, and $G * \mathbb{Z}$ acts faithfully on its normal form boundary $\partial L \subseteq X^\omega$. This boundary ∂L is a closed rational set in X^ω, and by Theorem 1.7, elements of $G * \mathbb{Z}$ act by rational homeomorphisms of ∂L, so $G * \mathbb{Z}$ embeds into the group $\mathcal{R}_{\partial L}$. It is easy to prove that ∂L has no isolated points in the case of a free product (see Corollary 2.11), so by Theorem 1.6 the group $\mathcal{R}_{\partial L}$ is isomorphic to the rational group defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ. $\square$

As far as we know, this theorem provides the first known connection between the theory of automatic groups and the theory of automata groups and rational homeomorphisms.

As mentioned earlier, constructing embeddings of hyperbolic groups into the rational group was a key step in constructing embeddings of hyperbolic groups into finitely presented simple groups [BBM21, BBMZ]. Here we have embedded each continuous automatic group G into the group of rational homeomorphisms of its normal form boundary ∂L (modulo taking a free product with $\mathbb{Z}$). This has a corresponding topological full group $[\![G \mid \partial L]\!]$, and the main question is under what circumstances this topological full group is finitely presented. Whenever it is finitely presented, such a group ought to admit an embedding into a finitely presented, simple twisted Brin–Thompson group, as it does for hyperbolic groups.

1.2. **Applications to specific groups.** We give three main examples of continuous automatic groups. First, we prove the following theorem in Section 2.2.

Theorem 1.10. *Every Baumslag–Solitar group $BS(m, n)$ is continuous asynchronous automatic, and thus embeds into the rational group.*

We prove this theorem using the asynchronous automatic structure on $BS(m, n)$ given in [ECH$^+$92, Section 7.4]. The resulting normal form boundary is the same as the ends of the group, i.e. the boundary of the corresponding Bass-Serre tree, and in particular $BS(m, n)$ acts faithfully on the normal form boundary. The embedding of $BS(m, n)$ into the rational group here is new, though embeddings of solvable Baumslag–Solitar groups $BS(1, n)$ into the rational group were previously obtained by Bartholdi and Sunik [Bv06]. Note that embeddings of the groups $BS(m, n)$ into finitely presented simple groups have already been obtained by Bux, Llosa Isenrich, and Wu using the action on the Bass–Serre tree [BIW25]. Indeed, their proof can be modified to show that the the topological full group for $BS(m, n)$ acting on its normal form boundary is finitely presented, and hence $BS(m, n)$ embeds into a corresponding twisted Brin–Thompson group.

Second, we prove the following theorem in Section 2.3.

Theorem 1.11. *Every Artin group A of finite type is continuous automatic, and hence embeds into the rational group.*

As a special case, this proves that braid groups embed into the rational group. We prove this theorem using the Deligne normal form for Artin groups of finite type [Del72]. This normal form was previously used by Charney [Cha92] to prove

that Artin groups of finite type are biautomatic, generalizing a result of Thurston for braid groups [ECH$^+$92, Chapter 9]. We show that the Deligne normal form is also continuous, and thus gives a continuous automatic structure. The resulting normal form boundary ∂L is very natural, and does not seem to have previously appeared in the literature. Our symbolic encoding gives ∂L the structure of a shift of finite type, and the group A acts faithfully on ∂L as long as A is not cyclic.

Even for braid groups, we do not know whether the corresponding topological full group $[\![A \mid \partial L]\!]$ is finitely presented. Embeddings of braid groups and some other Artin groups into finitely presented simple groups were obtained by the first author, Francesco Fournier-Facio, James Hyde, and Matthew Zaremsky in [BFFHZ25], but it is an open problem whether exceptional Artin groups of finite type (E_6, E_7, E_8, F_4, and H_4) admit embeddings into finitely presented simple groups.

Question 1.12. Does every Artin group of finite type admit an embedding into a finitely presented simple group?

Finally, recall that a group G is **cubulated** if it acts geometrically (i.e. properly and cocompactly by isometries) on a CAT(0) cube complex. We prove the following in Section 3.

Theorem 1.13. *Every virtually torsion-free cubulated group is continuous automatic, and thus embeds into the rational group.*

This theorem uses an automatic structure described by Niblo and Reeves [NR98], which uses "normal cube paths" in the corresponding CAT(0) cube complex. We refer to the corresponding tree of diagonal edges as the **normal diagonal tree**. The normal form boundary of the Niblo and Reeves automatic structure is the same as the (Gromov) boundary of the normal diagonal tree, and this turns out to be the same as another well-known boundary.

Theorem 1.14. *If X is a CAT(0) cube complex and v is a vertex of X, then the boundary of the normal diagonal tree in X based at v is naturally homeomorphic to the Roller boundary of X.*

Roller boundaries of CAT(0) cube complexes were introduced in [Rol98].

Any cubulated group G is finitely presented, but again we do not know whether the topological full group for G acting on the Roller boundary of the corresponding CAT(0) cube complex is finitely presented.

Question 1.15. Does every virtually torsion-free cubulated group admit an embedding into a finitely presented simple group?

Finally, we mention that many other interesting classes of groups are known to be automatic, and in each case it would be interesting to determine whether the automatic structure is continuous. For example, Mosher [Mos95] has proven that mapping class groups of finite type surfaces are automatic. Note that the Boone–Higman conjecture is open for mapping class groups of closed surfaces of genus three or greater.

Question 1.16. Are mapping class groups of finite-type surfaces continuous automatic? Do all such groups admit embeddings into finitely presented simple groups?

2. Examples of continuous automatic groups

In this section we prove our main criterion for a normal form to be continuous (Proposition 1.3), and then consider our three main example classes of continuous automatic groups: Baumslag–Solitar groups, Artin groups of finite type, and virtually torsion-free cubulated groups. We also prove several closure properties for continuous automatic groups.

2.1. **Criterion for continuity.** Given a group G and language L of normal forms for G, we write $|g \wedge h|_L$ for the length of the greatest common prefix of the normal forms for g and h. In particular, $|g \wedge g|_L$ is the length of the normal form for g.

Proposition 2.1. *Let G be a group, let $L \subseteq X^*$ be a language of normal forms for G, and let S be any subset of G that generates G as a monoid. Then L is continuous if and only if for every $s \in S$ and $N \in \mathbb{N}$ there exists an $M \in \mathbb{N}$ so that*

$$|g \wedge h|_L \geq M \quad \Rightarrow \quad |sg \wedge sh|_L \geq N$$

for all $g, h \in G$.

Proof. Since S generates G as a monoid, L is continuous if and only if the natural left action of each $s \in S$ on L extends to a continuous function $L \cup \partial L \to L \cup \partial L$.

Put a metric ρ (actually an ultrametric) on $X^{\leq \omega} = X^* \cup X^\omega$ by defining the distance between two distinct words to be 2^{-n}, where n is the length of the greatest common prefix of the words. Note that the balls in this metric are precisely the sets of words that have a given finite prefix, which are basic open sets in $X^{\leq \omega}$, and hence the metric topology determined by ρ is the same as the usual topology on $X^{\leq \omega}$. This metric ρ restricts to a metric on $L \cup \partial L$. Since $L \cup \partial L$ is compact and L is dense in $L \cup \partial L$, a function $f \colon L \to L$ extends to a continuous function $\overline{f} \colon L \cup \partial L \to L \cup \partial L$ if and only if f is uniformly continuous on L with respect to ρ (see [Bou98, §II.3.6, Theorem 2]).

The condition in the statement of the proposition certainly implies that the action of each $s \in S$ is uniformly continuous on L, and hence extends to a homeomorphism of $L \cup \partial L$. For the converse, suppose L is continuous, and let $s \in S$ and $N \in \mathbb{N}$. Then s acts uniformly continuously on L, so there exists a $\delta > 0$ such that

$$(1) \qquad\qquad \rho(\alpha, \beta) < \delta \quad \Rightarrow \quad \rho(s \cdot \alpha, s \cdot \beta) < 2^{-N}$$

for all $\alpha, \beta \in L$, where $s \cdot \alpha$ denotes the action of s on α. Let

$$k = \max\{|s^{-1} \cdot \alpha| \mid \alpha \in L \text{ and } |\alpha| < N\},$$

where $|\alpha|$ denotes the length of a word α. Let $M = 1 + \max(k, \lceil -\log_2 \delta \rceil)$, and suppose $g, h \in G$ and $|g \wedge h|_L \geq M$. If $g \neq h$, then the normal forms α and β for g and h satisfy $\rho(\alpha, \beta) = 2^{-|g \wedge h|_L} \leq 2^{-M} < \delta$, so $\rho(s \cdot \alpha, s \cdot \beta) < 2^{-N}$, and hence $|sg \wedge sh|_L \geq N$. If $g = h$, then the normal form for g has length greater than k, so the normal form for sg must have length N or greater, and hence $|sg \wedge sg|_L \geq N$. $\quad\square$

Remark 2.2. Note that Proposition 2.1 does *not* hold if S is merely a generating set for G as a group. Even if the elements of S acts uniformly continuously on L, the resulting continuous extensions $L \cup \partial L \to L \cup \partial L$ might not be bijections. For example, if L_3 is the normal form for a free abelian group of rank two given in Example 1.1, then x^{-1}, y, and y^{-1} all act as uniformly continuous bijections of L_3, but the action of x^{-1} extends to a non-injective map on $L_3 \cup \partial L_3$.

2.2. **Baumslag–Solitar groups.** Recall that the **Baumslag–Solitar groups** are the family of groups

$$BS(m, n) = \langle x, y \mid x^m y = y x^n \rangle$$

where $m, n \geq 1$.

Theorem 2.3. *The Baumslag–Solitar group $BS(m, n)$ is continuous asynchronous automatic for all $m, n \geq 1$.*

Proof. It is proven in [ECH$^+$92, Section 7.4] that $BS(m, n)$ has an asynchronous automatic structure (X, L), where $X = \{x, y, x^{-1}, y^{-1}\}$ and L is a normal form consisting of all freely reduced words of the form

$$x^a y^{\epsilon_1} x^{b_1} \cdots y^{\epsilon_k} x^{b_k}$$

where $k \geq 0$, $a \in \mathbb{Z}$, each $\epsilon_i = \pm 1$, and each b_i satisfies:

$$\epsilon_i = -1 \;\Rightarrow\; 0 \leq b_i < m \qquad \text{and} \qquad \epsilon_i = 1 \;\Rightarrow\; 0 \leq b_i < n.$$

We will prove that L is continuous using Proposition 2.1. Note that we can choose M separately for each element of X.

For the left action of x or x^{-1}, let $N \in \mathbb{N}$, let $M = N+1$, and let $g, h \in BS(m, n)$ with $|g \wedge h|_L \geq M$. The left action of $x^{\pm 1}$ on L simply replaces the initial x^a of a normal form with $x^{a \pm 1}$, so it follows easily that $|x^{\pm 1} g \wedge x^{\pm 1} h| \geq M - 1 = N$.

For the left action of y, let $N \in \mathbb{N}$, let $K = Nn/m + 2n$, and let

$$M = \max\left(K + \max(m, n), \; N + \left(\frac{K}{n} + 1\right)|m - n| + 2m \right).$$

Let $g, h \in G$ so that $|g \wedge h|_L \geq M$. We must prove that $|yg \wedge yh|_L \geq N$.

The left action of y on L works as follows. If we write a as $qn + r$ for some $q \in \mathbb{Z}$ and $0 \leq r < n$, then $yx^a = yx^{qn+r} = x^{qm} y x^r$ in $BS(m, n)$. Then the left action of y on L consists of replacing the initial x^a by $x^{qm} y x^r$, and then freely reducing the resulting word. Such reductions occur if and only if $r = 0$ and $\epsilon_1 = -1$, in which case we must freely reduce $x^{qm} y y^{-1} x^{b_1}$, where qm might be negative.

Since $M \geq K + \max(m, n)$, there are two cases:

(1) The normal forms for g and h start with the same $x^a y^{\epsilon_1} x^{b_1}$, where $|a| \leq K$.
(2) The normal forms for g and h start with the same x^a, where $|a| \geq K$

In the first case, note first that

$$|q| \leq \frac{|a|}{n} + 1 \leq \frac{K}{n} + 1.$$

When we replace $x^a = x^{qn+r} y$ by $x^{qm} y x^r$, the length of this prefix reduces by at most $|qm - qn|$, and then the subsequent cancellation reduces the length by at most $2m$. It follows that the normal forms for yg and yh share a common prefix of length at least

$$M - |qm - qn| - 2m \geq M - \left(\frac{K}{n} + 1\right)|m - n| - 2m \geq N$$

so $|yg \wedge yh|_L \geq N$. In the second case, note first that

$$|q| \geq \frac{|a|}{n} - 1 \geq \frac{K}{n} - 1$$

In this case, since $b_1 < m$ when $\epsilon_i = -1$, at most m letters from the word x^{qm} can cancel, so the normal forms for yg and yh will both start with the power of x of length at least

$$|qm| - m \geq \left(\frac{K}{n} - 1\right)m - m = N,$$

and therefore $|yg \wedge yh|_L \geq N$.

The argument for the left action of y^{-1} is similar, with the roles of m and n reversed. By Proposition 2.1, we conclude that L is continuous, so $BS(m, n)$ is continuous asynchronous automatic. $\square$

2.3. **Artin groups of finite type.** Given constants $m_{ij} \in \{2, 3, \ldots, \infty\}$ for $1 \leq i < j \leq n$, the associated **Artin group** is

$$A = \langle a_1, \ldots, a_n \mid \underbrace{a_i a_j a_i \cdots}_{m_{ij}} = \underbrace{a_j a_i a_j \cdots}_{m_{ij}} \text{ for all } i, j \rangle,$$

where the two products are alternating words of length m_{ij}, and there is no relation between a_i and a_j whenever $m_{ij} = \infty$. The elements $a_1, \ldots, a_n$ are called **Artin generators**. An Artin group has **finite type** if the quotient (Coxeter) group W obtained by adding the relations $a_i^2 = 1$ for all i is finite. The goal of this subsection is to prove the following theorem.

Theorem 2.4. *Every Artin group A of finite type with at least two generators has a continuous automatic structure L such that A acts faithfully on ∂L. Thus every such A embeds into the asynchronous rational group.*

In particular, this theorem holds for braid groups, which correspond to $m_{i,i+1} = 3$, and $m_{ij} = 2$ for $j \geq i + 2$. The proof of this theorem occupies the remainder of this section.

Charney [Cha92] proved that Artin groups of finite type are automatic (indeed biautomatic), generalizing a proof for braid groups due to Thurston [ECH+92]. This proof was later generalized to the class of Garside groups by Dehornoy [Deh02]. Charney's proof uses a normal form due to Deligne [Del72], which we will now describe.

If A is an Artin group of finite type, the submonoid A^+ generated by the Artin generators is called the **positive submonoid**. This monoid is finitely presented, with the relations being exactly the same as those for the Artin group given above . Since all the relations in A^+ equate words of the same length, the length function $x \mapsto |x|$ is a well-defined homomorphism $A^+ \to \mathbb{N}$.

There is a natural partial order on A^+ defined by $x \leq y$ if x is a left divisor of y, i.e. if $xz = y$ for some $z \in A^+$. The Artin generators have a unique least upper bound Δ in this partial order, known as the **Garside element**. This has the following properties:

(1) A^+ is generated by the set $(1, \Delta]$ of nontrival left divisors of Δ.

(2) For each nontrivial $x \in A^+$, there is a unique maximal element $\mathrm{pref}(x) \in (1, \Delta]$ for which $\mathrm{pref}(x) \leq x$.

(3) The square Δ^2 lies in the center of A, and conjugation by Δ is an involution that permutes the Artin generators.

(4) For every $x \in A$ there exists a $j \in \mathbb{N}$ such that $\Delta^j x \in A^+$.

We will describe the Deligne normal form using the generating set $(1, \Delta]$ for A as opposed to using the Artin generators. First, given any $x \in A^+$, the **left-greedy normal form** for x is the word $x_1 \cdots x_n$ over $(1, \Delta]$ for which $x_1 = \mathrm{pref}(x)$, and then $x_2 = \mathrm{pref}(x_1^{-1}x)$ if the remainder $x_1^{-1}x$ is nontrivial, and so forth. Note that this process terminates since $|x|$ is finite and each $|x_i| > 0$. Note also that it follows from property (3) that conjugation by Δ permutes the generating set $(1, \Delta]$, and therefore all of the Δ's in a left-greedy normal form must lie at the beginning of the word.

The **Deligne normal form** for an element $x \in A$ is the word $\Delta^{-j} x_1 \cdots x_n$, where j is the smallest non-negative integer for which $\Delta^j x \in A^+$, and $x_1 \cdots x_n$ is the left-greedy normal form for $\Delta^j x$. Charney gave a complete characterization of the language L of all Deligne normal forms. If $x, y \in (1, \Delta]$, we write $x \to y$ if $\mathrm{pref}(xy) = x$. Charney proved that

$$L = \{ \Delta^{-j} x_1 \cdots x_n \mid x_1 \to x_2 \to \cdots \to x_n \text{ and } x_1 \neq \Delta \text{ if } j > 0 \}.$$

It follows that the boundary ∂L is a subshift of finite type (indeed, a topological Markov chain) over the alphabet $(1, \Delta] \cup \{\Delta^{-1}\}$, with the allowed pairs being xy whenever $x \to y$, as well as $\Delta^{-1}\Delta^{-1}$ and $\Delta^{-1}x$ for all $x \in (1, \Delta)$.

Proof of Theorem 2.4. Charney proved that the Deligne normal form L is an automatic structure for A. We claim that L is a continuous normal form, and that the action of A on ∂L is faithful.

We begin by recalling the action of the generators $(1, \Delta] \cup \{\Delta^{-1}\}$ on L as described by Charney. First, the action of Δ^{-1} on L consists of prepending of Δ^{-1} and then freely reducing, and it follows easily that

$$|x \wedge x'|_L \geq N + 1 \quad \Rightarrow \quad |\Delta^{-1}x \wedge \Delta^{-1}x'|_L \geq N$$

for all $N \in \mathbb{N}$ and $x, x' \in A$. As for elements $y \in (1, \Delta]$, the left-action of y works as follows:

(1) If $x_1 \cdots x_n$ is a left-greedy normal form, then $y \cdot (x_1 \cdots x_n)$ starts with $z_1 = \mathrm{pref}(yx_1)$. Furthermore, there exists a $y' \in [1, \Delta]$ so that $yx_1 = z_1 y'$, and $y \cdot (x_1 \cdots x_n)$ is the concatenation of z_1 with $y' \cdot (x_2 \cdots x_n)$.

(2) If $\Delta^{-j} x_1 \cdots x_n$ is a Deligne normal form with $j > 0$, then there exists a $y' \in (1, \Delta]$ so that $y\Delta^{-j} = \Delta^{-j}y'$, and $y \cdot (\Delta^{-j} x_1 \cdots x_n)$ is obtained by freely reducing the concatenation of Δ^{-j} with $y' \cdot (x_1 \cdots x_n)$.

The free reduction in case (2) only occurs if $y' \cdot (x_1 \cdots x_n)$ begins with some power of Δ, in which case some word $\Delta^{-k}\Delta^k$ must be canceled. It follows from case (1) that

$$|x \wedge x'|_L \geq N \quad \Rightarrow \quad |yx \wedge yx'|_L \geq N$$

whenever $x, x' \in A^+$, but the case where $x, x' \in A \setminus A^+$ is more complicated.

For this remaining case, the key observation is that if $x_1 \cdots x_n$ is a left-greedy normal form and $x_1 \neq \Delta$, then at most $|y|$ of the generators at the beginning of $y \cdot (x_1 \cdots x_n)$ can be Δ. This is because $y \cdot (x_1 \cdots x_n)$ is the concatenation of some $z_1 \in (1, \Delta]$ with some $y' \cdot (x_2 \cdots x_n)$, where $yx_1 = z_1 y'$. Since $|y| + |x_1| = |z_1| + |y'|$ and $|z_1| > |x_1|$ whenever $z_1 = \Delta$, we must have $|y'| < |y|$ whenever $z_1 = \Delta$, so the result follows by induction.

We now claim that

$$|x \wedge x'|_L \geq N + 2|y| \quad \Rightarrow \quad |yx \wedge yx'|_L \geq N$$

for all $N \in \mathbb{N}$ and $x, x' \in A \setminus A^+$. To prove this, observe that if the Deligne normal forms for x and x' both start with $\Delta^{-N-|y|}$, then the Deligne normal forms for yx and yx' both start with Δ^{-N}, and hence $|yx \wedge yx'|_L \geq N$. If instead the Deligne normal forms for x and x' start with $\Delta^{-j}x_1 \cdots x_k$ for some $j < N + |y|$ and $k = (N + 2|y|) - j > |y|$, then the Deligne normal forms for yx and yx' both start with $y \cdot (\Delta^{-j}x_1 \cdots x_k)$, which has length at least $j + k - 2|y| > N$, and therefore $|yx \wedge yx'|_L \geq N$.

We conclude that L is a continuous normal form for A. Finally, to prove that the action of A on ∂L is faithful, let a_i and a_j be any two distinct Artin generators. It is well-known that $z = a_i a_j$ lies in $(1, \Delta)$, but $a_j^2 \notin (1, \Delta]$. In particular, $\mathrm{pref}(a_j^2) = a_j$, so $a_j \to a_j$ and hence $a_j^\infty = a_j a_j a_j \cdots$ is a point in ∂L. But $a_i \cdot a_j^\infty = z a_j^\infty$, and therefore a_i acts nontrivially on ∂L. $\square$

2.4. **Closure properties.**

Definition 2.5. An (asynchronous) automatic structure (X, L) on a group G is **continuous** if L is a continuous language of normal forms. If such a structure exists, we say that G is **continuous (asynchronous) automatic**, and the set ∂L is its **rational boundary**.

We will prove in Proposition 4.8 that ∂L is, in fact, a closed rational subset of X^ω.

Example 2.6 (Finite groups). Any finite group G is continuous automatic. Indeed, if L is any language of normal forms for G, then $\partial L = \emptyset$, so the action of G on L extends continuously to the action of G on $L \cup \partial L$.

Example 2.7 (Free groups). Let F_n be a free group with free generating set S. Let $X = S \cup S^{-1}$, and let $L \subseteq X^*$ be the language of all reduced words over S, including the empty word. Then L is a language of normal forms for F_n and (X, L) is an automatic structure. The rational boundary ∂L is the usual Gromov boundary for F_n, and the action of F_n on L extends continuously to $L \cup \partial L$ in the usual way, so F_n is continuous automatic.

Proposition 2.8. *If G and H are continuous automatic or continuous asynchronous automatic groups, then so is $G \times H$.*

Proof. Let (X, L) and (Y, M) be continuous automatic or continuous asynchronous automatic structures on G and H, respectively. We may assume that L and M both contain the empty string. We regard $G \times H$ as generated by the disjoint union $X \cup Y$, where elements of X commute with elements of Y. It is proven in [ECH$^+$92, Theorems 4.1.1 and 7.3.5] that $(X \cup Y, LM)$ is an automatic or asynchronous automatic structure for $G \times H$, where $LM = \{\alpha\beta \mid \alpha \in L \text{ and } \beta \in M\}$. Note that LM is a normal form for $G \times H$, so it suffices to prove that LM is continuous.

Observe first that $\partial(LM) = \partial L \cup L(\partial M)$, where $L(\partial M)$ denotes all concatenations of finite words in L and infinite words in ∂M. The natural homeomorphism $L \times M \to LM$ extends to a quotient map

$$q \colon (L \cup \partial L) \times (M \cup \partial M) \to LM \cup \partial(LM)$$

where q maps $L \times \partial M$ homeomorphically to $L(\partial M) \subseteq \partial(LM)$, and q maps $\partial L \times (M \cup \partial M)$ to $\partial L \subseteq \partial(LM)$ via projection onto the first factor. The continuous action of $G \times H$ on $(L \cup \partial L) \times (M \cup \partial M)$ permutes the fibers of q, and therefore descends to a continuous action of G on $LM \cup \partial(LM)$, as desired. $\square$

We will need the following theorem about free products, mostly for the purpose of obtaining a Cantor space on which a given continuously automatic group G acts faithfully.

Theorem 2.9. *If G and H are continuous automatic groups, then the free product $G * H$ is continuous automatic. Indeed, as long as $|G| \geq 3$ and $|H| \geq 2$, the group $G * H$ has a rational boundary ∂L which is a Cantor space, and $G * H$ acts faithfully on ∂L.*

Before proving it, we need the following result.

Lemma 2.10. *Let G be a continuous automatic group. Then there exists a continuous automatic structure (X, L) on G such that L contains the empty word.*

Proof. Let (X, L) be any continuous automatic structure on G. Then L is a normal form for G, so there exists a word $\eta \in L$ that represents the identity. Let $L' = (L \setminus \{\eta\}) \cup \{\varepsilon\}$. Then (X, L') is also an automatic structure for G (see [BGSS92, Corollary II.B.2.5]), and $\partial L' = \partial L$, so (X, L') is a continuous automatic structure. $\square$

Proof of Theorem 2.9. Let (X_G, L_G) and (X_H, L_H) be continuous automatic structures on G and H. By Lemma 2.10, we may assume that L_G and L_H both contain the empty word. Let X be the disjoint union $X_G \cup X_H$, and let L be the language consisting of the empty word together with all finite alternating products of words in $L_G \setminus \{\varepsilon\}$ and words in $L_H \setminus \{\varepsilon\}$. Then L is a language of nomal forms for $G * H$ and by [BGSS92, Theorem A] the pair (X, L) is an automatic structure for $G * H$. Thus it suffices to prove that L is continuous, its boundary ∂L is a Cantor space, and the action of $G * H$ on ∂L is faithful.

For continuity we use Proposition 2.1. Let $x \in X$ and $N \in \mathbb{N}$. Since $x \in X_G \cup X_H$, we can suppose without loss of generality that $x \in X_G$. Since L_G is a continuous language of normal forms for G, by Proposition 2.1 there exists an $M \in \mathbb{N}$ so that
$$|g \wedge g'|_{L_G} \geq M \quad \Rightarrow \quad |xg \wedge xg'|_{L_G} \geq N$$
for all $g, g' \in G$. Let $M' = \max(1, M + N)$, let $u, u' \in G * H$, and suppose that $|u \wedge u'|_L \geq M'$. We wish to prove that $|xu \wedge xu'|_L \geq N$. Since $M' > 0$, neither u nor u' can be the identity, so they are both alternating words in $G \setminus \{1\}$ and $H \setminus \{1\}$. Since $M' \geq M$, we can consider the following three cases:

(1) Both u and u' start with the same element of $G \setminus \{1\}$, whose normal form in L_G has length less than M.

(2) Both u and u' start with elements of $G \setminus \{1\}$ whose normal forms in L_G both have length M or greater.

(3) Both u and u' start with elements of $H \setminus \{1\}$.

In the first case, suppose u and u' both start with some $g \in G \setminus \{1\}$ whose normal form has length less than M. Then $|g^{-1}u \wedge g^{-1}u'|_L \geq M' - M \geq N$, so $|xu \wedge xu'|_L = |xg \wedge xg|_{L_G} + |g^{-1}u \wedge g^{-1}u'|_L \geq N$. In the second case, suppose u and u' start with elements $g, g' \in G \setminus \{1\}$, where $|g \wedge g'|_{L_G} \geq M$. Then $|xg \wedge xg'|_{L_G} \geq N$, and it follows that $|xu \wedge xu'|_L \geq N$. Finally, if u and u' both start with elements of $H \setminus \{1\}$, then $|xu \wedge xu'|_L \geq |u \wedge u'|_L \geq M \geq N$, as desired. By Proposition 2.1, we conclude that L is continuous.

For the rest suppose $|G| \geq 3$ and $|H| \geq 2$. The boundary ∂L has two different types of words:

(1) Infinite alternating products of words from $L_G \setminus \{\varepsilon\}$ and $L_H \setminus \{\varepsilon\}$.
(2) Words which are the product of a finite word from L and an infinite word from ∂L_G or ∂L_H.

No words of type (1) are isolated points, and each word of type (2) is a limit of word of type (1), so ∂L has no isolated points. Since ∂L is a closed subset of the Cantor space X^ω, it follows from Brouwer's theorem that ∂L is itself a Cantor space. For faithfulness, recall that every nontrivial element of $G * H$ is an alternating product of elements of $G \setminus \{1\}$ and $H \setminus \{1\}$. Any such element must act nontrivially on some of the words of type (1): those that end with an element of $G \setminus \{1\}$ act nontrivially on words with a prefix in $L_H \setminus \{\varepsilon\}$, and similarly those that end in an element of $H \setminus \{1\}$ act nontrivially on words with a prefix in $L_G \setminus \{\varepsilon\}$. We conclude that the action of $G * H$ on ∂L is faithful. □

Corollary 2.11. *Let G be a continuous automatic group. Then G embeds in a continuous automatic group H with rational boundary ∂L such that ∂L is a Cantor space and H acts faithfully on ∂L.*

Proof. By Theorem 2.9, the group $H = \mathbb{Z} * G$ suffices as long as G is nontrivial (and $H = F_2 * G$ suffices even for trivial G). □

2.5. **Extension to groupoids.** In this section we extend the results so far to include groupoids with finitely many objects. This extension will be necessary in Section 3 to prove that torsion-free cubulated groups are continuously automatic.

Recall that a **groupoid** G is a small category in which every morphism has an inverse. We think of G as the set of morphisms, together with source and target functions $s, t \colon G \to \mathrm{Obj}(G)$ and partially defined binary operation, where $g_1 g_2$ is defined if and only if $s(g_1) = t(g_2)$.

A groupoid is **connected** if every pair of objects has at least one morphism between them. The **isotropy group** of a groupoid G at an object $v \in \mathrm{Obj}(G)$ is the group of all morphisms from v to v. If a groupoid G is connected, then the isotropy groups at any two objects of G are isomorphic.

A **finite category generating set** for a groupoid G is any finite set X of morphisms such that every morphism in G is a finite composition of those in X. Note that if a groupoid G has a finite category generating set X, then G must have finitely many objects. Since arbitrary compositions of elements of X are not defined, there is no canonical surjection $X^* \twoheadrightarrow G$. Instead, there is a regular language $X^{\mathrm{allowed}} \subset X^+$ consisting of all allowed compositions of finitely many elements of X, and a canonical surjection $\pi \colon X^{\mathrm{allowed}} \twoheadrightarrow G$. A language $L \subseteq X^*$ is called a **language of normal forms** for G if $L \subseteq X^{\mathrm{allowed}}$ and L maps bijectively to G under π.

An **automatic structure** on a groupoid G is a pair (X, L), where X is a finite category generating set for G, and $L \subseteq X^*$ is a regular language contained in X^{allowed} such that $\pi(L) = G$ and for each $x \in X$, the relation

$$R_x = \{(\alpha, \beta) \in L \times L \mid x\alpha \in X^{\mathrm{allowed}} \text{ and } \pi(\beta) = \pi(x\alpha)\}$$

is synchronous rational as a subset of $X^* \times X^*$. A groupoid G is **automatic** if it has an automatic structure. If we instead require that the relations R_x are deterministic rational, then G is **asynchronous automatic**. As with groups, it is possible to prove that any (asynchronous) automatic groupoid G has an (asynchronous)

automatic structure (X, L) for which L is a language of normal forms. (The proof in [ECH$^+$92, Theorem 7.3.2] goes through without much modification.)

Recall that an **action** of a groupoid G on a set A consists of a function $o\colon A \to \mathrm{Obj}(G)$ and a partially defined function $G \times A \to A$, where $g \cdot a$ is defined if and only if $s(g) = o(a)$, such that $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G$ and $a \in A$ for which $g_1 g_2$ and $g_2 \cdot a$ are defined.

Suppose X is a finite category generating set for a groupoid G, and L is a language of normal forms for G, with canonical bijection $\pi\colon L \to G$. Then we can define a function $o\colon L \to \mathrm{Obj}(G)$ by $o(\alpha) = t(\pi(\alpha))$. This determines a natural action of G on L by $g \cdot \alpha = \pi^{-1}(g\,\pi(\alpha))$. The function o extends continuously to $L \cup \partial L$, and we say that the language of normal forms L is **continuous** if the action of G on L extends continuously to an action of G on $L \cup \partial L$. A groupoid G is **continuous (asynchronous) automatic** if there exists an (asynchronous) automatic structure (X, L) on G for which L is a continuous language of normal forms.

The following is our main theorem about groupoids. The version of this theorem without the word "continuous" is used in [NR98] without proof, though the proof is very similar to the proofs of [ECH$^+$92, Proposition 11.1.3].

Theorem 2.12. *Let G be a connected groupoid and let $v \in \mathrm{Obj}(G)$. If G is continuous (asynchronous) automatic then the isotropy group of G at v is continuous (asynchronous) automatic.*

Proof. Let (X, L) be a continuous (asynchronous) automatic structure for G. Let G_v denote the isotropy group of G at v, and let

$$L_v = \{\alpha \in L \mid \pi(\alpha) \in G_v\}.$$

Note that L_v is a regular language, being the set of all words $x_1 \cdots x_m$ in the regular language L for which $s(x_m) = t(x_1) = v$. It follows that $L_v \times L_v$ is synchronous rational .

Since X is finite, G must have finitely many objects, say $\mathrm{Obj}(G) = \{v_1, \ldots, v_n\}$. Since G is connected, we can choose elements $t_1, \ldots, t_n \in G$ such that $s(t_i) = v$ and $t(t_i) = v_i$ for each i. Let $\rho\colon G \to G_v$ be the function

$$\rho(g) = t_{t(g)}^{-1} g t_{s(g)}.$$

Note that ρ restricts to the identity on G_v and ρ is a homomorphism, i.e. $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ whenever $g_1 g_2$ is defined. Assuming G_v is infinite, it is always possible to modify our choices of the t_i's so that ρ is injective on X. Let $X_v = \rho(X)$, and let $\rho_*\colon X^* \to X_v^*$ be the induced bijection. We claim that $(X_v, \rho_*(L_v))$ is a continuous (asynchronous) automatic structure on G_v.

Note first that $\rho_*(L_v)$ is indeed a language of normal forms for G_v. Since L is continuous, we know that the action of G on L extends continuously to an action of G on $L \cup \partial L$. Since L_v is invariant under the action of G_v, it follows that G_v acts continuously on $L_v \cup \partial L_V$. But $\rho_*\colon X^* \to X_v^*$ is an isometry, and the restriction $\rho_*\colon L_v \to \rho_*(L_v)$ is G_v-equivariant, so it follows that G_v acts continuously on $\rho_*(L_v) \cup \partial\rho_*(L_v)$. This proves that $\rho_*(L_v)$ is a continuous language of normal forms for L_v.

All that remains is to prove that $(X_v, \rho_*(L_v))$ is an (asynchronous) automatic structure on G_v. Note first that $\rho_*(L_v)$ is a regular language since L_v is. We wish

to prove that the relation

$$R_{G_v,\rho(x)} = \{(\alpha,\beta) \in \rho_*(L_v) \times \rho_*(L_v) \mid \pi(\beta) = \pi(\rho(x)\alpha)\} \subset X_v^* \times X_v^*$$

is synchronous rational (or deterministic rational) for each $x \in X$. We know that

$$R_{G,g} = \{(\alpha,\beta) \in L \times L \mid s(g) = t(\pi(\alpha)) \text{ and } \pi(\beta) = g\,\pi(\alpha)\}$$

is synchronous rational (or deterministic rational) for each $g \in G$. But since ρ is finite-to-one, we can write

$$R_{G_v,\rho(x)} = (\rho_* \times \rho_*)\left((L_v \times L_v) \cap \bigcup_{g \in \rho^{-1}(\rho(x))} R_{G,g}\right)$$

and this is synchronous rational (or deterministic rational) since the class of such relations is closed under finite unions and intersections. $\square$

3. CAT(0) CUBE COMPLEXES

The goal of this section is to prove that groups acting freely and cocompactly on CAT(0) cube complexes embed into the rational group $\mathcal{R}_2$. In particular, we want to exploit Theorem 2.12 and Corollary 5.8. Hence, we just need to show that the Niblo–Reeves automatic structure [NR98] is continuous. In the process of doing so, we describe an important connection between the rational boundary and the well-known Roller boundary [Rol98, BCG$^+$09].

3.1. **Roller boundary.** Recall that a **midcube** of the unit cube $[0,1]^n$ is any subspace obtained by setting one of the coordinates equal to $1/2$. A connected subspace of a CAT(0) cube complex is called a **hyperplane** if its intersection with each cube C is either empty or is a midcube of C. Every edge of a CAT(0) cube complex crosses exactly one hyperplane, and each hyperplane separates a CAT(0) cube into two connected components known as **half-spaces**. See [Sch23, Chapter 5] for further background on hyperplanes and half-spaces.

Let Y be a CAT(0) cube complex, and let $\mathcal{H}(Y)$ be the set of all hyperplanes in Y. For each hyperplane $H \in \mathcal{H}(Y)$, let $h_0(H)$ and $h_1(H)$ denote the two corresponding half-spaces, where the assignment of 0 and 1 is arbitrary. Given a vertex $v \in Y^0$, we can define a function $\delta_v \colon \mathcal{H}(Y) \to \{0,1\}$ by

$$\delta_v(H) = \begin{cases} 0 & \text{if } v \in h_0(H), \\ 1 & \text{if } v \in h_1(H). \end{cases}$$

Then the mapping $v \mapsto \delta_v$ gives an embedding of the discrete space Y^0 into the Cantor space $\{0,1\}^{\mathcal{H}(Y)}$. The topological boundary of Y^0 in $\{0,1\}^{\mathcal{H}(Y)}$ is known as the **Roller boundary** of Y, denoted ∂Y.

3.2. **Normal diagonal paths.** A **diagonal edge** in a CAT(0) cube complex Y is any line segment that connects two opposite corners of a cube in Y. Note that each edge of Y is also a diagonal edge, since it connects two opposite corners of a 1-cube. A path of diagonal edges in Y is called a **diagonal path**, and the sequence of cubes crossed by a diagonal path is called a **cube path** in the literature. We will prefer to deal with diagonal paths rather than cube paths.

If $v \in Y^0$, we say that a hyperplane H is **adjacent** to v if there exists an edge emanating from v that crosses H. A diagonal path with vertices $v_0, \ldots, v_n$ is **normal** if for each i, the diagonal edge from v_i to v_{i+1} crosses precisely the set of
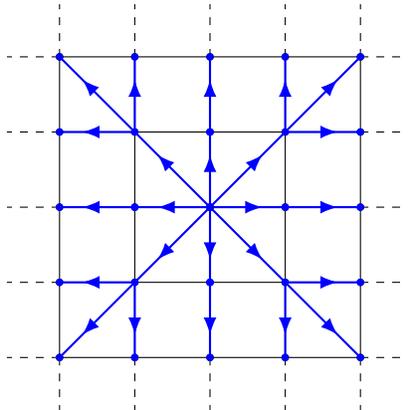
FIGURE 1. The normal diagonal tree for $\mathbb{Z}^2$.

all hyperplanes that are adjacent to v_i and separate v_i from v_n. We will use the following facts about normal diagonal paths:

(1) If v and w are vertices of Y, there exists a unique normal diagonal path from v to w.
(2) A diagonal path with vertices $v_0, \ldots, v_n$ is normal if and only if each subpath v_i, v_{i+1}, v_{i+2} consisting of two consecutive diagonal edges is normal. In particular, every subpath of a normal diagonal path is normal.
(3) A normal diagonal path crosses each hyperplane at most once.

See [NR98, Section 3] for proofs (phrased using normal cube paths). If v is a vertex of Y, it follows from the statements above that the union of all normal diagonal paths starting at v is a rooted tree of diagonal edges that contains all of the vertices of Y. We will refer to this as the **normal diagonal tree** rooted at v.

The following theorem does not seem to have previously appeared in the literature.

Theorem 3.1. *Let Y be a locally finite* CAT(0) *cube complex, let $v \in Y^0$, and let $\mathcal{T}_v$ be the normal diagonal tree rooted at v. Then the inclusion $Y^0 \hookrightarrow \mathcal{T}_v$ induces a homeomorphism from the Roller boundary ∂Y to the Gromov boundary $\partial \mathcal{T}_v$.*

Proof. We will prove that there exists a continuous injection $h \colon Y^0 \cup \partial \mathcal{T}_v \to Y^0 \cup \partial Y$ which restricts to the identity on Y^0. Since the domain and codomain are both compact Hausdorff and Y^0 is dense in both, it follows that h is a homeomorphism.

To define h, observe that any infinite path in $\mathcal{T}_v$ starting at the root crosses each hyperplane in Y at most once, since all of its finite prefixes are normal diagonal paths. It follows that the sequence of vertices on such a path converges to a point in ∂Y, and this determines a mapping $h \colon Y^0 \cup \partial \mathcal{T}_v \to Y^0 \cup \partial Y$.

To prove that h is continuous, we claim that each hyperplane $H \in \mathcal{H}(Y)$ is crossed by only finitely many edges of $\mathcal{T}_d$. This follows from a basic fact of CAT(0) cubical geometry: if w is a vertex and H is a hyperplane which is not adjacent to w, then there exist only finitely many hyperplanes H' that separate w from H, and at least one of these is adjacent to w. In particular, if there are n hyperplanes that separate the base vertex v from a hyperplane H, then any normal diagonal path emanating from v that crosses H must do so after at most $n+1$ steps. We

conclude that all edges of $\mathcal{T}_d$ that cross H lie in the first n levels of the tree, so there are finitely many such edges.

Now, the space $\{0,1\}^{\mathcal{H}(Y)}$ has a subbasis for its topology consisting of all points that assign either a 0 or a 1 to a given hyperplane H. By the results of the previous paragraph, the preimage of each of these sets under h is clopen in $Y^0 \cup \partial\mathcal{T}_d$, and therefore h is continuous.

Finally, to prove that h is injective, consider any pair of distinct infinite paths α, α' in $\mathcal{T}_v$ starting at the root, which pass through vertices $\{w_n\}$ and $\{w'_n\}$, respectively. Let k be the maximum integer for which $w_k = w'_k$, so w_{k+1} and w'_{k+1} are distinct vertices connected to w_k by diagonal edges. Let H be a hyperplane that separates w_{k+1} from w'_{k+1}, and suppose without loss of generality that w_k is on the same side of H as w_{k+1}, with w'_{k+1} on the opposite side. Now, for all $n > k$, the subpath of α from w_k to w_n is a normal diagonal path, as is the subpath of α' from w_k to w'_n. Since H is adjacent to w_k and the diagonal edge from w_k to w_{k+1} does not cross H, all of the vertices w_n for $n > k$ lie on the same side of H as w_k. But since the diagonal edge from w_k to w'_{k+1} crosses H, all of the vertices w'_n for $n > k$ lie on the opposite side of H. It follows that the sequences $\{w_n\}$ and $\{w'_n\}$ do not converge to the same point in ∂Y, which proves that h is injective. $\square$

3.3. **Continuous automaticity.** If Z is a topological space and $B \subseteq Z$, recall that the **fundamental groupoid** of Z based at B, denoted $\pi_1(Z, B)$, is the groupoid whose objects are the points of B, with a morphism from z_1 to z_2 being a homotopy class of paths in Z from z_1 to z_2. The isotropy groups of this groupoid are the fundamental groups $\pi_1(Z, z)$ for different points $z \in B$.

Let G be a group acting freely and cocompactly on a finite dimensional CAT(0) cube complex Y, and let $\mathcal{G}$ be the fundamental groupoid $\pi_1(Y/G, Y^0/G)$. Let $\kappa \colon Y \to Y/G$ be the quotient map. A **directed cube** is a cube in the complex Y with two (ordered) diagonally opposite vertices specified. Each directed cube as an associated geodesic diagonal path in Y from one vertex to another, the uniqueness of such a path is provided by the fact that Y is CAT(0). Define $\mathbb{A}$ to be set of homotopy classes of projection of diagonal paths in Y/G. Directed cubes in Y can be labeled by $\mathbb{A}$, note that two different directed cubes starting from the same vertex are labeled differently thanks to the unique lifting property. For any $x \in Y^0$, let $L_{\kappa(x)}$ be the subset of $\mathbb{A}^*$ given by the labels of normal cube-paths in $\mathcal{C}_x$. Note that $L_{\kappa(x)}$ depends only on $\kappa(x)$, and not on the chosen vertex $x \in Y^0$, since the trees of normal cube-paths starting at different points of Y that lie in the same G-orbit are isomorphic. It follows that we have a bijective map

$$\pi_{\mathcal{G}} : \bigcup_{\kappa(x) \in Y^0/G} L_{\kappa(x)} \to \mathcal{G}.$$

Moreover, Niblo and Reeves [NR98] prove that $\mathcal{G}$ has a natural automatic structure $\bigcup_{\kappa(x) \in Y^0/G} L_{\kappa(x)}$.

Proposition 3.2. *Let G be a group acting freely and cocompactly on a finite dimensional CAT(0) cube complex Y, and let $\mathcal{G}$ be the fundamental groupoid $\pi_1(Y/G, Y^0/G)$. Then $\mathcal{G}$ is continuously automatic.*

Proof. We must prove that the automatic structure on $\mathcal{G}$ is continuous.

If $x, y \in Y^0$, let $\mathcal{G}(\kappa(x), \kappa(y))$ denote all the elements of $\mathcal{G}$ which are homotopy classes of paths from $\kappa(x)$ to $\kappa(y)$. There is a natural right action of $\mathcal{G}$ on Y/G by

monodromy, which we denote by $x \cdot \gamma$ for $x \in Y^0$ and $\gamma \in \mathcal{G}$. Note that $x \cdot \gamma$ is only defined if $\gamma \in \mathcal{G}(\kappa(x), \kappa(y))$ for some $y \in Y^0$.

By Theorem 3.1, each $x \in Y^0$ determines a bijection $\phi_x \colon L_{\kappa(x)} \to Y^0$ by $\phi_x(\alpha) = x \cdot \pi_{\mathcal{G}}(\alpha)$. By Theorem 3.1, ϕ_x always extends to a homeomorphism $\phi_x \cup \Phi_x \colon L_{\kappa(x)} \cup \partial L_{\kappa(x)} \to Y^0 \cup \partial Y$. Note that the left action of G commutes with the right action of $\mathcal{G}$, i.e. $g(x \cdot \gamma) = g(x) \cdot \gamma$ for all $x \in Y^0$, $g \in G$, and $\gamma \in \mathcal{G}$ for which $x \cdot \gamma$ is defined. It follows that $g \circ \phi_x = \phi_{g(x)}$ for all $x \in Y^0$ and $g \in G$, and hence $g \circ \Phi_x = \Phi_{g(x)}$ for all $x \in Y^0$ and $g \in G$.

Each $\gamma \in \mathcal{G}(\kappa(x), \kappa(y))$ determines a mapping $L_{\kappa(x)} \to L_{\kappa(y)}$ by $\gamma \cdot \alpha = \pi_{\mathcal{G}}^{-1}(\gamma \, \pi_{\mathcal{G}}(\alpha))$, and we want to prove that these mappings extends continuously to the boundaries of the languages. But observe that

$$
\begin{aligned}
\phi_{x \cdot \gamma^{-1}}^{-1}(\phi_x(\alpha)) &= \phi_{x \cdot \gamma^{-1}}^{-1}(x \cdot \pi_{\mathcal{G}}(\alpha)) \\
&= \phi_{x \cdot \gamma^{-1}}^{-1}\big((x \cdot \gamma^{-1}) \cdot (\gamma \, \pi_{\mathcal{G}}(\alpha))\big) = \pi_{\mathcal{G}}^{-1}\big(\gamma \, \pi_{\mathcal{G}}(\alpha)\big) = \gamma \cdot \alpha
\end{aligned}
$$

That is, γ maps $L_{\kappa(x)}$ to $L_{\kappa(y)}$ via $\phi_{x \cdot \gamma^{-1}}^{-1} \circ \phi_x$. Both $\phi_x \colon L_{\kappa(x)} \to Y^0$ and $\phi_{x \cdot \gamma^{-1}} \colon L_{\kappa(y)} \to Y^0$ extend continuously to boundary homeomorphisms $\Phi_x \colon \partial L_{\kappa(x)} \to \partial Y$ and $\Phi_{x \cdot \gamma^{-1}} \colon \partial L_{\kappa(y)} \to \partial Y$, so the action of γ mapping $L_{\kappa(x)} \to L_{\kappa(y)}$ extends continuously to $\Phi_{x \cdot \gamma^{-1}}^{-1} \circ \Phi_x \colon \partial L_{\kappa(x)} \to \partial L_{\kappa(y)}$. $\qquad \square$

Combining Proposition 3.2 with Theorem 2.12 and Corollary 5.8, we have the desired result.

Theorem 3.3. *A group acting freely and cocompactly on a finite dimensional CAT(0) cube complex embeds into the asynchronous rational group $\mathcal{R}_2$.*

Theorem 3.1 together with the discussion above it say that the rational embedding of groups acting on CAT(0) cube complexes and hyperbolic groups are achieved by their actions on the horofunction boundary in both cases.

4. Automata

In this section we introduce the class of continuous automatic groups and define the rational boundary of such a group. We extend the definition of rational homeomorphisms to arbitrary closed rational sets, and we prove that any continuous asynchronous automatic group acts by rational homeomorphisms on its rational boundary. Finally, we use this to prove that any such group embeds into the asynchronous rational group for the full shift defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ.

Our arguments use several different kinds of automata, all of which are either *acceptors* that accept finite words, or *path automata* that accept infinite words. We begin by defining acceptors, automatic groups, continuous automatic groups, and their rational boundaries. We then define path automata and prove that the rational homeomorphisms of a closed rational set are precisely those homeomorphisms whose graphs are accepted by deterministic path automata, from which all of our main results follow.

4.1. **Automata for finite words.** By a **finite word**, we mean any finite sequence of symbols from some finite alphabet X. We let X^* denote the set of all finite words over X, i.e. the free monoid generated by X. This includes the identity element ε, which we call the **empty word**. Any subset of X^* is called a **language**.
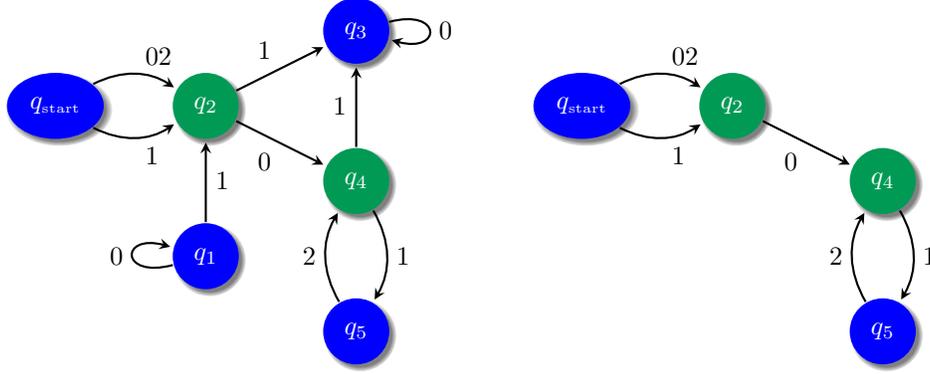
FIGURE 2. On the left, an acceptor over $\{0, 1, 2\}^*$ with six states and ten transitions, where $Q_{\text{accept}} = \{q_2, q_4\}$. On the right, an equivalent trim acceptor. Both acceptors accept the language $L = \{1, 02, 020(12)^n, 10(12)^n \mid n \in \mathbb{N}\}$, and neither is deterministic.

4.1.1. *Acceptors over X^*.* If X is a finite alphabet, an **acceptor** over X^* is a tuple $A = (Q, T, q_{\text{start}}, Q_{\text{accept}})$, where:

   (1) Q is a finite set whose elements are called **states**.
   (2) T is a finite subset of $Q \times X^* \times Q$ whose elements are called **transitions**.
   (3) $q_{\text{start}} \in Q$ is a specified **start state**.
   (4) $Q_{\text{accept}} \subseteq Q$ is a finite set of **accept states**.

We can regard A as a finite directed graph (see e.g. Figure 2) whose vertices are the states and whose edges are the transitions, where each transition (q_1, α, q_2) becomes a directed edge $q_1 \xrightarrow{\alpha} q_2$ labeled by a word $\alpha \in X^*$. A **path** in A is a finite sequence of transitions

$$q_0 \xrightarrow{\alpha_1} q_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} q_n$$

and the corresponding **label** is the concatenation $\alpha_1 \alpha_2 \cdots \alpha_n$. We say that A **accepts** an word $\alpha \in X^*$ if there exists a path in A that begins at the start state, ends at an accept state, and has label α. The set of all such words is the **regular language accepted by A**, and two acceptors over X^* are **equivalent** if they accept the same regular language.

If X and Y are finite alphabets, an **acceptor over $X^* \times Y^*$** is defined similarly, but the transitions are labeled by elements of the Cartesian product $X^* \times Y^*$. In this case, the label for a path

$$q_0 \xrightarrow{(\alpha_1, \beta_1)} q_1 \xrightarrow{(\alpha_2, \beta_2)} \cdots \xrightarrow{(\alpha_n, \beta_n)} q_n$$

is the pair $(\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n)$. The set of all pairs in $X^* \times Y^*$ accepted by such an acceptor is called a **rational relation**.

An acceptor (over X^* or $X^* \times Y^*$) is **trim** if every state lies on at least one path from the start state to an accept state. Every acceptor that accepts at least a word or pair of words is equivalent to a trim acceptor, obtained by removing all non-start states and transitions that do not lie on such a path.

4.1.2. *Deterministic acceptors over X^*.* An acceptor over X^* is **deterministic** if every transition is labeled by a single letter from X, and no two transitions from the

same state have the same label. The following proposition is well known, but we include its proof for the sake of completeness. The construction of the automaton A' in the proof is known as the **powerset construction**.

Proposition 4.1. *Every regular language $L \subseteq X^*$ is accepted by some deterministic acceptor.*

Proof. First observe that we can accept L with an acceptor A whose transitions are labeled by elements of $X \cup \{\varepsilon\}$. Specifically, such an A can be obtained from any acceptor that accepts L by replacing each transition $q \xrightarrow{x_1 x_2 \cdots x_n} q'$ labeled by a word of length $n \geq 2$ by a path $q \xrightarrow{x_1} q_1 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} q_{n-1} \xrightarrow{x_n} q'$, where $q_1, \ldots, q_{n-1}$ are new non-accept states.

Given such an A, we will define a new acceptor $A' = (Q', T', q'_{\text{start}}, Q'_{\text{accept}})$ which is equivalent to A, but is deterministic. For each $\alpha \in X^*$, let

$$Q(\alpha) = \{q \in Q \mid \text{there is a path in } A \text{ from } q_{\text{start}} \text{ to } q \text{ labeled } \alpha\}.$$

We let $Q' = \{Q(\alpha) \mid \alpha \in X^*\}$, so each state of Q' is a subset of Q, of which there are finitely many. Let $q'_{\text{start}} = Q(\varepsilon)$, and let Q'_{accept} be the collection of all sets in Q' that contain an accept state. Finally, let T' be the set of all transitions of the form $Q(\alpha) \xrightarrow{x} Q(\alpha x)$ for $\alpha \in X^*$ and $x \in X$. Note that $Q(\alpha x) = Q(\beta x)$ whenever $Q(\alpha) = Q(\beta)$, so there is exactly one transition starting from each state in Q' labeled by each element of X. Then A' is deterministic, and it accepts exactly the same words as A. $\qquad\square$

4.1.3. *Deterministic acceptors over $X^* \times Y^*$.* The definition of a deterministic acceptor over $X^* \times Y^*$ is slightly more involved. First we introduce a new letter $\dashv$ called the **endmarker**, and we consider the expanded alphabets $X_\dashv = X \cup \{\dashv\}$ and $Y_\dashv = Y \cup \{\dashv\}$. A **deterministic acceptor** over $X^* \times Y^*$ is an acceptor A over $X_\dashv^* \times Y_\dashv^*$ that has the following properties:

(1) For every state q in A, either all of the transitions from q are labeled by distinct elements of $X_\dashv \times \{\varepsilon\}$, or all of the transitions from q are labeled by distinct elements of $\{\varepsilon\} \times Y_\dashv$. States of the first type are called **X-states**, and states of the second type are called **Y-states**.

(2) Every element of $X_\dashv^* \times Y_\dashv^*$ accepted by A has the form $(\alpha\dashv, \beta\dashv)$ for some $(\alpha, \beta) \in X^* \times Y^*$.

If A is a deterministic acceptor, we refer to the set

$$R = \{(\alpha, \beta) \in X^* \times Y^* \mid A \text{ accepts } (\alpha\dashv, \beta\dashv)\}$$

as the **deterministic rational relation** accepted by A. Note that R is indeed a rational relation, namely the relation accepted by the acceptor A' over $X^* \times Y^*$ obtained from A by replacing all $(\dashv, \varepsilon)$ and $(\varepsilon, \dashv)$ labels by $(\varepsilon, \varepsilon)$ labels.

A **synchronous acceptor** over $X^* \times Y^*$ is a deterministic acceptor such that every finite path starting at the start state has transition labels that alternate between $X \times \varepsilon$ and $\varepsilon \times Y$ until the first endmarker is reached. (This is equivalent to the usual definition of a synchronous acceptor involving labels in $X \times Y$ and padding. See [ECH$^+$92, Lemma 7.1.2]). A relation $R \subseteq X^* \times Y^*$ is **synchronous rational** if it is accepted by some synchronous acceptor.

Example 4.2. If $X = \{0\}$ and $Y = \{1\}$, then the relation

$$\{(0^n, 1^{2n}) \mid n \geq 0\}$$

is deterministic rational but not synchronous rational, while the relation

$$\{(0^n, 1^{2n}) \mid n \geq 0\} \cup \{(0^{2n}, 1^n) \mid n \geq 0\}$$

is rational but not deterministic rational.

4.1.4. *Automatic groups.* If G is a group, a **monoid generating set** for G is any set X that generates G as a monoid. For example, any group generating set for G which is closed under inversion is a monoid generating set for G. If X is a monoid generating set for G, there is a canonical surjection $\pi \colon X^* \to G$ that interprets any word over X as an element of G.

An **automatic structure** on a group G is a pair (X, L), where X is a finite monoid generating set for G and $L \subseteq X^*$ is a regular language such that $\pi(L) = G$ and such that for each $x \in X$, the relation

$$R_x = \{(\alpha, \beta) \in L \times L \mid \pi(\beta) = \pi(x\alpha)\}$$

is synchronous rational as a subset of $X^* \times X^*$. A group G is **automatic** if it has an automatic structure.

If we instead require that the relations R_x are deterministic rational instead of synchronous rational, then (X, L) is called an **asynchronous automatic structure** on G. A group G is **asynchronous automatic** if it has an asynchronous automatic structure. Every automatic group is asynchronous automatic, but the converse does not hold. In particular, Thurston proved that the Baumslag–Solitar group $BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$ is asynchronous automatic for all m and n, but is only automatic if $m = n$ (see [ECH$^+$92, Section 7.4]).

Remark 4.3. It also makes sense to consider pairs (X, L) for which the relations R_x are only required to be rational. Such a pair is called a "quasi-automatic structure", and a group that admits such a structure is qa "quasi-automatic group". Blanchette has recently proven that a group is quasi-automatic if and only if it is asynchronous automatic [Bla19].

If G is a group and X is a monoid generating set for G, a language $L \subseteq X^*$ is called a **language of normal forms** for G if the canonical surjection $\pi \colon X^* \to G$ maps L bijectively to G. The following theorem is proven in [ECH$^+$92, Theorem 7.3.2].

Theorem 4.4. *Let G be an asynchronous automatic group and let X be a finite monoid generating set for G. Then there exists a language of normal forms $L \subseteq X^*$ such that (X, L) is an asynchronous automatic structure for G.*

Example 4.5. Let $G = \langle x, y \mid xy = yx \rangle \cong \mathbb{Z} \times \mathbb{Z}$, and let $X = \{x, y, x^{-1}, y^{-1}\}$. Then the set

$$L = \{x^m y^n \mid m, n \in \mathbb{Z}\}$$

is a language of normal forms for G. See Figure 3 for the deterministic acceptor of L and Figure 4 for the synchronous relation

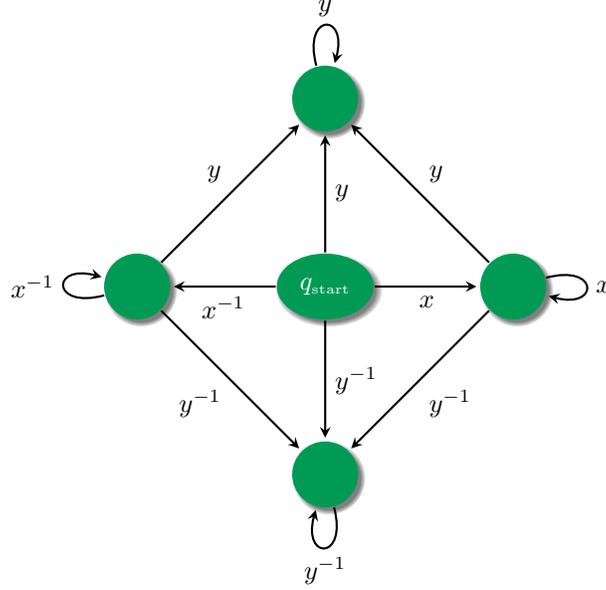$$R_y = \{(x^m y^n, x^m y^{n+1}) \mid m, n \in \mathbb{Z}\}.$$

FIGURE 3. The deterministic acceptor for the normal form in Example 4.5. Note that all the states are in Q_{accept}.

4.2. **Infinite words and continuous automatic structures.** If X is a finite alphabet, an **infinite word** over X is an infinite sequence $x_1 x_2 x_3 \cdots$ of elements of X. The set X^ω of all infinite words can be given the infinite product topology, and for $|X| \geq 2$ it is homeomorphic to the Cantor set by a well-known result of Brouwer (see e.g. [Wil70, Theorem 30.3]).

There is also a natural topology on the set $X^{\leq \omega} = X^* \cup X^\omega$, where a basic open set consists of all words that have a given finite prefix. Under this topology, a sequence $\{\alpha_n\}$ of finite words converges to an infinite word β if and only if each finite prefix of β is also a prefix of all but finitely many of the α_n's. This space $X^{\leq \omega}$ is compact and metrizable, with each point of X^* being an isolated point.

4.2.1. *Continuous normal forms.* If $L \subseteq X^*$ is a language, the **boundary** of L is the set ∂L of accumulation points of L in X^ω. That is, an infinite word $\beta \in X^\omega$ lies in ∂L if and only if every finite prefix of β is also a prefix of some element of L.

Definition 4.6. Let G be a group, and let X be a finite monoid generating set for G. We say that a language $L \subseteq X^*$ of normal forms is **continuous** if the natural action of G on L extends continuously to an action of G on $L \cup \partial L$.

Here the natural action of G on L refers to the action for which each $g \in G$ maps each $\alpha \in L$ to the unique word $\beta \in L$ satisfying $\pi(\beta) = g\,\pi(\alpha)$.

Example 4.7. Let G be the group $\langle x, y \mid xy = yx \rangle \cong \mathbb{Z} \times \mathbb{Z}$, and let $X = \{x, y, x^{-1}, y^{-1}\}$. Then the language

$$L = \{x^m y^n \mid x, y \in \mathbb{Z}\}$$

is a continuous language of normal forms for G. In particular, ∂L consists of the infinite words $x^\infty = xxx \cdots$ and $x^{-\infty} = x^{-1} x^{-1} x^{-1} \cdots$, as well as the words $x^m y^\infty$
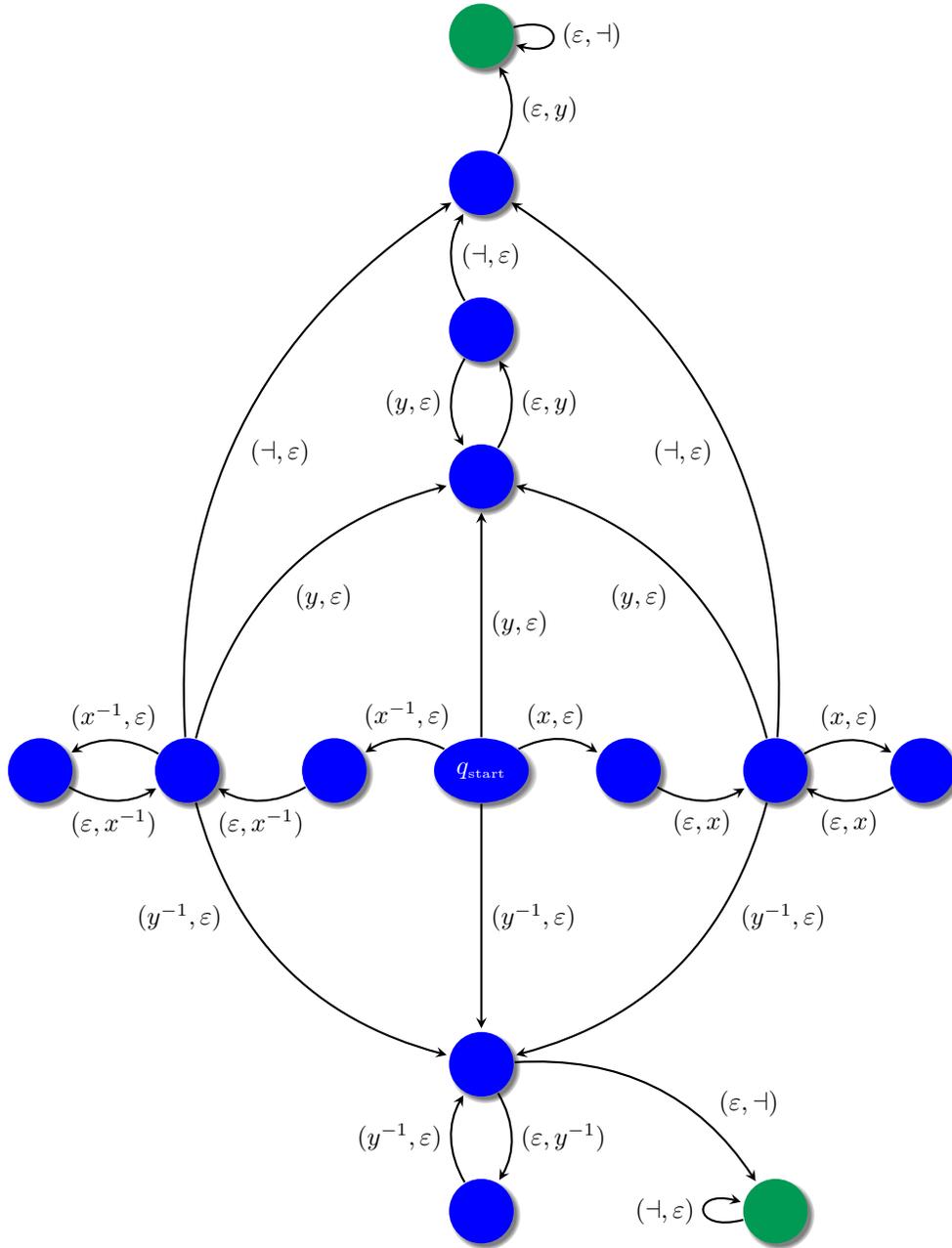
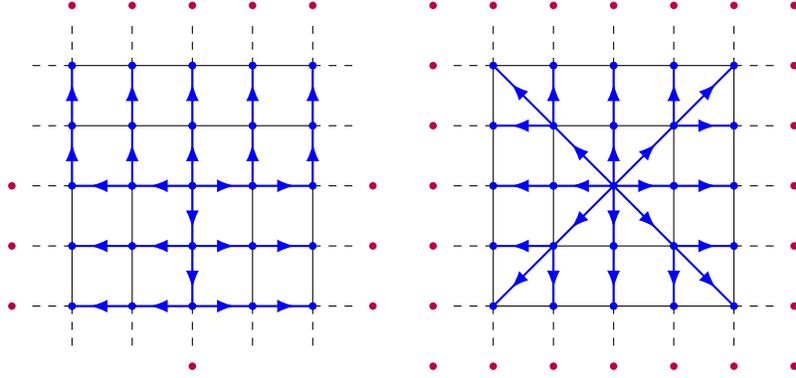FIGURE 4. The synchronous acceptor for the relation R_y in Example 4.5. Here the accept states are in green.

FIGURE 5. On the left, a graphic interpretation of a portion of $L' \cup \partial L'$. On the right, the same for $L'' \cup \partial L''$, note that $x^{\pm 1} y^{\pm 1}$ are treated as single diagonal arrows for simplicity.

and $x^m y^{-\infty}$ for all $m \in \mathbb{Z}$. The induced action of G on ∂L is not faithful, with the element $y \in G$ acting trivially on ∂L, while the element $x \in G$ fixes x^∞ and $x^{-\infty}$ and maps each $x^m y^{\pm\infty}$ to $x^{m+1} y^{\pm\infty}$.

On the other hand,

$$L' = \{x^m y^n \mid m \in \mathbb{Z} \text{ and } n \geq 0\} \cup \{y^{-n} x^m \mid m \in \mathbb{Z} \text{ and } n > 0\}.$$

is a language of normal forms for G which is not continuous. Here the boundary $\partial L'$ consists of the words

$$x^\infty, \quad x^{-\infty}, \quad x^m y^\infty, \quad y^{-n} x^\infty, \quad y^{-n} x^{-\infty}, \quad y^{-\infty}$$

for $m \in \mathbb{Z}$ and $n > 0$, and the natural action of G on L' does not extend continuously to $\partial L'$. For example, the sequences x^k and $x^k y$ both converge to x^∞ as $k \to \infty$, but left-multiplying by y^{-1} gives the sequences $y^{-1} x^k$ and x^k of words in L', which converge to $y^{-1} x^\infty$ and x^∞, respectively.

Incidentally, there does exist a continuous language L'' of normal forms for G such that G acts faithfully on $\partial L''$, namely the language consisting of all words

$$(xy)^m x^n, \ (xy)^m y^n, \ (xy^{-1})^m x^n, \ (xy^{-1})^m y^{-n},$$
$$(x^{-1} y)^m x^{-n}, \ (x^{-1} y)^m y^n, \ (x^{-1} y^{-1})^m x^{-n}, \ (x^{-1} y^{-1})^m y^{-n}$$

for $m, n \geq 0$. See Figure 5 for a graphic interpretation of L' and L'' together with their boundaries.

4.3. **Automata for infinite words.** So far we have seen how to use automata to accept languages of finite words, as well as relations between such languages. We now define a new kind of automaton to accept families of infinite words. First, observe that if $\{\alpha_n\}$ is a sequence of words in X^*, then the infinite concatenation $\alpha_1 \alpha_2 \cdots$ lies in $X^{\leq \omega} = X^* \cup X^\omega$, with the product lying in X^* if and only if all but finitely many of the α_n's are empty.

4.3.1. *Path automata.* By a **path automaton** over X^*, we mean a triple $A = (Q, T, q_{\text{start}})$, where Q is a finite set of states, $T \subseteq Q \times X^* \times Q$ is a finite set of

transitions, and $q_{\mathrm{start}} \in Q$ is the start state. A path automaton has no specified set of accept states. An **infinite path** in A is a sequence of transitions

$$q_0 \xrightarrow{\alpha_1} q_1 \xrightarrow{\alpha_2} q_3 \xrightarrow{\alpha_3} \cdots$$

and the corresponding **label** is the infinite product $\alpha_1 \alpha_2 \alpha_3 \cdots$. We say that A is **nondegenerate** if, for every infinite path, its label $\alpha_1 \alpha_2 \alpha_3 \cdots$ has infinitely many α_i's which are non-empty. In this case, we say that A **accepts** an element of X^ω if there exists an infinite path beginning at the start state that has the given label. The set of all elements of X^ω accepted by a path automaton is always closed in the product topology, and is called a **closed rational set**.

Path automata over $X^* \times Y^*$ are defined similarly, and such an automaton is nondegenerate if every infinite path has label in $X^\omega \times Y^\omega$. Such automata accept the closed rational subsets of $X^\omega \times Y^\omega$.

Two path automaton are **equivalent** if they accept the same closed rational set. A path automaton is **trim** if every state lies on an infinite path beginning at the start state. Any path automaton that accepts a nonempty closed rational set is equivalent to a trim path automaton, obtained by removing any states and transitions that do not lie on such a path.

Proposition 4.8. *If $L \subseteq X^*$ is a regular language, then ∂L is a closed rational set in X^ω.*

Proof. We may assume $L \neq \emptyset$, so by Proposition 4.1, there exists a trim, deterministic acceptor $A = (Q, T, q_{\mathrm{start}}, Q_{\mathrm{accept}})$ that accepts L. Let $A' = (Q, T, q_{\mathrm{start}})$ be the corresponding path automaton, and note that A' is nondegenerate since each transition is labeled by a single letter from X. Let $E \subseteq X^\omega$ be the closed rational set accepted by A'. We claim that $E = \partial L$.

Viewing the set T of transitions as an alphabet, we can consider the compact space $T^{\leq \omega} = T^* \cup T^\omega$ of all sequences of edges in A. This has a closed subspace $P \subseteq T^{\leq \omega}$ consisting of all finite paths in A from the start state to an accept state, together with all infinite paths in A beginning at the start state. Since A is trim, every infinite path in P is a limit of finite paths in P, so $P \cap T^*$ is dense in P. (It's important here that A be trim, so that every infinite path starting at the start state is a limit of finite paths that end at some accept state.)

Labeling determines a continuous function $\varphi \colon T^{\leq \omega} \to X^{\leq \omega}$, which satisfies $\varphi(P \cap T^*) = L$ and $\varphi(P \cap T^\omega) = E$. Since P is compact and $P \cap T^*$ is dense in P, it follows that $L \cup E$ is compact and L is dense in $L \cup E$. In particular, $L \cup E$ is precisely the closure of L in $X^{\leq \omega}$, so $E = (L \cup E) \cap X^\omega$ must be equal to ∂L. $\square$

It follows from this proposition that every closed rational subset of X^ω is the boundary of some regular language. In the language of symbolic dynamics, these are precisely the follower sets in sofic shifts.

Remark 4.9. Automata that accept infinite words have been studied thoroughly in [PP04]. Our notion of a closed rational set accepted by a path automaton is a special case of the usual notion of a rational set accepted by a Büchi automaton (see [PP04, Section I.5]). Here a Büchi automaton is the same as an acceptor, except that it is viewed as accepting all labels for infinite paths that begin at the start state and pass through accept states infinitely many times. Our path automata are simply Büchi automata for which every state is an accept state, and it is well

known that a rational set is closed if and only if it is determined by such a Büchi automaton [PP04, Theorem III.3.9].

4.3.2. *Deterministic path automata.* A path automaton A over $X^* \times Y^*$ is **deterministic** if for every state $q \in Q$, either all of the transitions from q are labeled by distinct elements of $X \times \{\varepsilon\}$, or all of the transitions from q are labaled by distinct elements of $\{\varepsilon\} \times Y$. As with deterministic acceptors, we refer to the first kind of state as an **X-state**, and the second kind of state as a **Y-state**. A closed set $R \subseteq X^\omega \times Y^\omega$ which is accepted by a nondegenerate deterministic path automaton is **deterministic rational**.

A deterministic path automaton is **synchronous** if every transition goes from either an X-state to a Y-state or from a Y-state to an X-state. The corresponding closed subsets of $X^\omega \times Y^\omega$ are called **synchronous rational**.

We also have the following analog of Proposition 4.8 for boundaries of regular languages.

Proposition 4.10. *Let X and Y be finite alphabets, let $L \subseteq X^*$ be a regular language, and let $f \colon L \cup \partial L \to Y^{\leq \omega}$ be a continuous function with $f(L) \subseteq Y^*$ and $f(\partial L) \subseteq Y^\omega$. If the graph of $f|_L$ is a rational relation determined by a trim acceptor $A = (Q, T, q_{start}, Q_{accept})$, then the corresponding path automaton $A' = (Q, T, q_{start})$ is nondegenerate and accepts the graph of $f|_{\partial L}$.*

Proof. Let $E \subseteq X^{\leq \omega} \times Y^{\leq \omega}$ be the set of all labels of infinite paths in A' beginning at the start state. As in the proof of Proposition 4.8, we can consider the compact space $T^{\leq \omega} = T^* \cup T^\omega$, and let $P \subseteq T^{\leq \omega}$ be the closed subspace consisting of all finite paths in A from the start state to an accept state, together with all infinite paths in A beginning at the start state. Since A is trim, we know that $P \cap T^*$ is dense in P.

Let $\varphi \colon T^{\leq \omega} \to X^{\leq \omega} \times Y^{\leq \omega}$ be the labeling map, so $\varphi(P \cap T^*)$ is the graph of $f|_L$ and $\varphi(P \cap T^\omega) = E$. Since P is compact and $P \cap T^*$ is dense in P, the image $\varphi(P)$ is compact and the graph of $f|_L$ is dense in $\varphi(P)$. But since f is continuous, the graph of f is closed in $T^{\leq \omega}$, and in particular the graph of f is the closure of the graph of $f|_L$. We conclude that $\varphi(P)$ is the graph of f, so E is equal to the graph of $f|_{\partial L}$. Since $f(\partial L) \subseteq Y^\omega$, we have that $E \subseteq X^\omega \times Y^\omega$, which proves that A' is nondegenerate, and A' accepts the graph of $f|_{\partial L}$. $\square$

Proposition 4.11. *Let X and Y be finite alphabets, let $L \subseteq X^*$ be a regular language, and let $f \colon L \cup \partial L \to Y^{\leq \omega}$ be a continuous function with $f(L) \subseteq Y^*$ and $f(\partial L) \subseteq Y^\omega$. If the graph of $f|_L$ is a deterministic rational relation in $X^* \times Y^*$, then the graph of $f|_{\partial L}$ is a deterministic closed rational set in $X^\omega \times Y^\omega$.*

Proof. Let $L\dashv = \{\alpha\dashv \mid \alpha \in L\}$, where $\dashv$ is the endmarker, and note that $\partial(L\dashv) = \partial L$. Let $f_\dashv \colon L\dashv \cup \partial L \to Y_\dashv^{\leq \omega}$ be the function that agrees with f on ∂L and maps each $\alpha\dashv \in L\dashv$ to $f(\alpha)\dashv$. Then $f_\dashv$ is continuous, $f_\dashv(L\dashv) \subseteq Y_\dashv^*$, and $f_\dashv(\partial L) \subseteq Y^\omega \subseteq Y_\dashv^\omega$.

Let $A = (Q, T, q_{\text{start}}, Q_{\text{accept}})$ be a trim deterministic acceptor over $X^* \times Y^*$ that accepts the graph of $f|_L$. Then A can be viewed as a trim acceptor over $X_\dashv^* \times Y_\dashv^*$ that accepts the graph of $f_\dashv|_{L\dashv}$. By Proposition 4.10, the path automaton $A' = (Q, T, q_{\text{start}})$ over $X_\dashv^* \times Y_\dashv^*$ is nondegenerate and accepts the graph of $f_\dashv|_{\partial(L\dashv)} = f|_{\partial L}$. Since the graph of $f|_{\partial L}$ is contained in $X^\omega \times Y^\omega$, no infinite path in A' beginning at the start state passes through a transition labeled $(\dashv, \varepsilon)$ or $(\varepsilon, \dashv)$. Then the

path automaton A'' over $X^* \times Y^*$ which is obtained from A' by removing all such transitions also accepts the graph of $f|_{\partial L}$. Note that A'' is deterministic since each state is an X-state or Y-state, and therefore the graph of $f|_{\partial L}$ is a deterministic closed rational set. $\qquad\square$

4.3.3. *Transducers.* A path automaton A over $X^* \times Y^*$ is called a **transducer** if it satisfies the following conditions:

(1) Every transition is labeled by an element of $X \times Y^*$.
(2) For each state q, no two distinct transitions starting at q involve the same letter of X.

If A is a nondegenerate transducer, then the subset of $X^\omega \times Y^\omega$ accepted by A must be the graph of a continuous function $f\colon E \to Y^\omega$, where the domain $E \subseteq X^\omega$ is a closed rational set. We refer to such a function as **rational map**. This generalizes the definition of rational maps on full shifts given in [GNS00], as well as the definition of rational maps on subshifts of finite type given in [BBMZ].

5. Rationality for continuous automatic groups

5.1. **Proof of the main theorem.** We are now in a position to prove our main theorem.

5.1.1. *Adding delay.* We begin by describing a construction that takes a deterministic path automaton $A = (Q, T, q_{\text{start}})$ over $X^* \times Y^*$ and defines an equivalent automaton $A^{(n)}$ over $X^* \times Y^*$ whose input in the second coordinate is delayed by n steps.

To define $A^{(n)}$ we need the following terminology. A word $\alpha \in X^*$ will be called an **input word** at a state $q \in Q$ if there exists a finite path in A beginning at q whose label has the form (α, β) for some $\beta \in Y^*$. Given an $n \geq 1$, the corresponding **delayed automaton** $A^{(n)}$ has the following states:

(S1) All pairs $(q_{\text{start}}, x_1 \cdots x_k)$, where $x_1 \cdots x_k$ is an input word at q_{start} of length less than n, and
(S2) All pairs $(q, x_1 \cdots x_n)$, where $q \in Q$ and $x_1 \cdots x_n$ is an input word at q of length n.

The start state of $A^{(n)}$ is $(q_{\text{start}}, \varepsilon)$, and $A^{(n)}$ has the following transitions:

(T1) One transition $(q_{\text{start}}, x_1 \cdots x_{k-1}) \xrightarrow{(x_k, \varepsilon)} (q_{\text{start}}, x_1 \cdots x_k)$ for each nonempty input word $x_1 \cdots x_k$ at q_{start} of length at most n.
(T2) One transition $(q, x_1 \cdots x_n) \xrightarrow{(x_{n+1}, \varepsilon)} (q', x_2 \cdots x_{n+1})$ for each transition in A of the form $q \xrightarrow{(x_1, \varepsilon)} q'$ and each input word $x_2 \cdots x_{n+1}$ at q' of length n.
(T3) One transition $(q, x_1 \cdots x_n) \xrightarrow{(\varepsilon, y)} (q', x_1 \cdots x_n)$ for each transition in A of the form $q \xrightarrow{(\varepsilon, y)} q'$ and each input word $x_1 \cdots x_n$ at q' of length n.

Proposition 5.1. *Let A be a nondegenerate, deterministic path automaton over $X^* \times Y^*$. Then for each $n \geq 1$, the delayed automaton $A^{(n)}$ is nondegenerate, deterministic, and is equivalent to A.*

Proof. Observe first that $A^{(n)}$ is indeed deterministic, with each state of type (S1) being an X-state, and each state $(q, x_1 \cdots x_n)$ of type (S2) being an X-state or Y-state depending on whether q is. In particular, each (S1) state has at most one outgoing (T1) transition for each element of X, while each (S2) state either has at

most one outgoing (T2) transition for each element of X, or at most one outgoing (T3) transition for each element of Y.

Now, every infinite path in $A^{(n)}$ beginning at $(q_{\text{start}}, \varepsilon)$ has the form

$$(q_{\text{start}}, \varepsilon) \xrightarrow{(x_1, \varepsilon)} (q_{\text{start}}, x_1) \xrightarrow{(x_2, \varepsilon)} (q_{\text{start}}, x_1 x_2) \xrightarrow{(x_3, \varepsilon)} \cdots$$

$$\cdots \xrightarrow{(x_n, \varepsilon)} (q_{\text{start}}, \gamma_0) \xrightarrow{(\alpha_1, \beta_1)} (q_1, \gamma_1) \xrightarrow{(\alpha_2, \beta_2)} (q_2, \gamma_2) \xrightarrow{(\alpha_3, \beta_3)} \cdots$$

for some $x_1, \ldots, x_n \in X$ and $q_i \in Q$, where each $\gamma_i \in X^*$ consists of the last n letters of $x_1 \cdots x_n \alpha_1 \cdots \alpha_i$. Such a path has the same label as the path

$$q_{\text{start}} \xrightarrow{(\mu_1, \nu_1)} q_1 \xrightarrow{(\mu_2, \nu_2)} q_2 \xrightarrow{(\mu_3, \nu_3)} \cdots$$

in A, where for each i we have $(\mu_i, \nu_i) = (\alpha_i, \beta_i)$ if $(\alpha_i, \beta_i) \in \{\varepsilon\} \times Y$, and $(\mu_i, \nu_i) = (x, \varepsilon)$ if $(\alpha_i, \beta_i) \in X \times \{\varepsilon\}$, where x is the first letter of γ_i. Conversely, given an infinite path in A as above, we can recover the corresponding path in $A^{(n)}$ by letting $x_1, \ldots, x_n$ be the first n letters of $\mu_1 \mu_2 \cdots$, letting γ_i be first n letters of $\mu_{i+1} \mu_{i+2} \cdots$, letting $(\alpha_i, \beta_i) = (\mu_i, \nu_i)$ if $(\mu_i, \nu_i) \in \{\varepsilon\} \times Y$, and letting $(\alpha_i, \beta_i) = (x, \varepsilon)$ if $(\mu_i, \nu_i) \in X \times \{\varepsilon\}$, where x is the last letter of γ_{i+1}. We conclude that the infinite paths in $A^{(n)}$ starting at $(q_{\text{start}}, \varepsilon)$ have exactly the same labels as the infinite paths in A starting at q_{start}, so $A^{(n)}$ is nondegenerate and equivalent to A. □

See Figure 6 and Figure 7 for a deterministic path automaton A and the corresponding delayed automaton $A^{(1)}$.

5.1.2. *Characterization of rational maps.* We are now ready to prove a characterization of rational maps.

Theorem 5.2. *Let X and Y be finite alphabets, let $E \subseteq X^\omega$ be a closed rational set (see definition in Section 4.3.1) and let $f \colon E \to Y^\omega$ be a continuous function. Then f is rational if and only if the graph of f is a deterministic closed rational set in $X^\omega \times Y^\omega$.*

We begin with the following lemma.

Lemma 5.3. *Let A be a nondegenerate, trim, deterministic path automaton over $X^* \times Y^*$. If A accepts the graph of some function $f \colon E \to Y^\omega$, where $E \subseteq X^\omega$, then there exists an $n \geq 1$ such that every Y-state in $A^{(n)}$ has exactly one outgoing transition.*

Proof. Let $A = (Q, T, q_{\text{start}})$. Given a state $q \in Q$, we call a word $\alpha \in X^*$ a **hedge word** at q if there exist finite paths beginning at q with labels (α, β) and (α, β') such that β and β' start with different letters. We claim that every state in Q has finitely many hedge words.

To prove this, let $q \in Q$, and suppose to the contrary that q has infinitely many distinct hedge words $\{\alpha_n\}$. Let $T^{\leq \omega} = T^* \cup T^\omega$ with labeling map $\varphi \colon T^{\leq \omega} \to X^{\leq \omega}$, and let $P \subseteq T^{\leq \omega}$ be the closed subspace of all finite or infinite paths starting at q. Since each α_n is a hedge word, we can find paths $\{\sigma_n\}$ and $\{\sigma_n'\}$ in $P \cap T^*$ so that $\varphi(\sigma_n) = (\alpha_n, \beta_n)$ and $\varphi(\sigma_n') = (\alpha_n, \beta_n')$ for each n, where β_n and β_n' start with different letters. Since P is compact, so is $P \times P$, which means that the set of all pairs (σ_n, σ_n') has an accumulation point $(\sigma, \sigma') \in P \times P$. Then σ and σ' are infinite paths starting at q with $\varphi(\sigma) = (\alpha, \beta)$ and $\varphi(\sigma') = (\alpha, \beta')$ for some $\alpha \in X^\omega$ and $\beta, \beta' \in Y^\omega$, where β and β' start with different letters. Since A is trim, there exists

a finite path σ_0 in A from q_{start} to q, which has some label (α_0, β_0). Then $\sigma_0 \sigma$ and $\sigma_0 \sigma'$ are infinite paths in A starting at q_{start} with labels $(\alpha_0 \alpha, \beta_0 \beta)$ and $(\alpha_0 \alpha, \beta_0 \beta')$, respectively. Since A accepts the graph of f, it follows that $\beta_0 \beta = f(\alpha_0 \alpha) = \beta_0 \beta'$, which is impossible since $\beta \neq \beta'$. We conclude that each state in Q has only finitely many hedge words.

Since Q is finite and each state in Q has only finitely many hedge words, there exists an $n \geq 1$ so that no state in Q has a hedge word of length n. Then every Y-state $(q, x_1 \cdots x_n)$ of $A^{(n)}$ has at most one outgoing transition $(q, x_1 \cdots x_n) \xrightarrow{(\varepsilon, y)} (q', x_1 \cdots x_n)$, where y is the unique first letter of all possible $\beta \in Y^*$ for which $(x_1 \cdots x_n, \beta)$ is the label of some finite path starting at q, and q' is the end state of the unique transition from q labeled (ε, y).   □

Proof of Theorem 5.2. Note that the graph of f is closed in $X^\omega \times Y^\omega$ since E is compact and f is continuous. If f is rational and A is a nondegenerate transducer for f, then we can construct a nondegenerate deterministic path automaton that accepts f by replacing each transition $q \xrightarrow{(x, y_1 \cdots y_n)} q'$ in A by a path

$$q \xrightarrow{(x, \varepsilon)} q_1 \xrightarrow{(\varepsilon, y_1)} \cdots \xrightarrow{(\varepsilon, y_{n-1})} q_{n-1} \xrightarrow{(\varepsilon, y_n)} q'$$

where $q_1, \ldots, q_{n-1}$ are new Y-states, and each original state q becomes an X-state.

For the converse, suppose that $A = (Q, T, q_{\text{start}})$ is a nondegenerate, deterministic path automaton that accepts the graph of f. By Lemma 5.3, there exists an $n \geq 1$ so that every Y-state in $A^{(n)}$ has at most one outgoing transition. Then every Y-state (q_0, α) in $A^{(n)}$ is the beginning of a unique path

$$(q_0, \alpha) \xrightarrow{(\varepsilon, y_1)} (q_1, \alpha) \xrightarrow{(\varepsilon, y_2)} \cdots \xrightarrow{(\varepsilon, y_k)} (q_k, \alpha)$$

where (q_k, α) is an X-state. If $(q, \beta) \xrightarrow{(x, \varepsilon)} (q_0, \alpha)$ is a transition from an X-state into (q_0, α), then removing this transition from $A^{(n)}$ and adding the transition $(q, \beta) \xrightarrow{(x, y_1 \cdots y_k)} (q_k, \alpha)$ yields an equivalent path automaton. Applying this change to every transition in $A^{(n)}$ from an X-state to a Y-state and then removing all of the Y-states, we obtain a path automaton equivalent to $A^{(n)}$ which is a transducer. Since $A^{(n)}$ is equivalent to A by Proposition 5.1, this transducer accepts the graph of f, so f is rational.   □

See Section 5.2 for an example which illustrates the proof. In particular, it describes a path automaton, then it computes a delayed version and retrieves a transducer.

5.1.3. *The rational group.* Theorem 5.2 gives a nice characterization of rational homeomorphisms which is completely symmetric between the domain and range. As far as we know, this result does not seem to have previously appeared in the literature, even in the case of rational homeomorphisms of full shifts. This immediately yields the following.

Proposition 5.4. *Let $E \subseteq X^\omega$ and $E' \subseteq Y^\omega$ be closed rational sets, and let $f \colon E \to E'$ be a rational homeomorphism. Then the inverse $f^{-1} \colon E' \to E$ is rational.*   □

The special case of Proposition 5.4 where E and E' are full shifts was previously shown in [GNS00, Proposition 2.21] , and versions for subshifts of finite type appeared in [BBM21, Proposition 2.14] and [BBMZ, Lemma 2.11]. The same sources also prove special cases of the following proposition.

Proposition 5.5. *Let X, Y, and Z be finite alphabets, and let $E \subseteq X^\omega$ and $E' \subseteq Y^\omega$ be closed rational sets. If $f\colon E \to E'$ and $g\colon E' \to Z^\omega$ are rational maps, then the composition $g \circ f$ is rational.*

Proof. Let $A = (Q, T, q_{\text{start}})$ and $A' = (Q', T', q'_{\text{start}})$ be nondegenerate trim transducers for f and g, respectively. Then a transducer A'' for $g \circ f$ can be constructed as follows. First, the states of A'' are all pairs $(q, q') \in Q \times Q'$, and the start state is $(q_{\text{start}}, q'_{\text{start}})$. For each transition $q_0 \xrightarrow{(x, y_1 \cdots y_n)} q_1$ in A and each path

$$q'_0 \xrightarrow{(y_1, \alpha_1)} q'_1 \xrightarrow{(y_2, \alpha_2)} \cdots \xrightarrow{(y_n, \alpha_n)} q'_n,$$

in A', there is one corresponding transition $(q_0, q'_0) \xrightarrow{(x, \alpha_1 \cdots \alpha_n)} (q_1, q'_n)$ in A''. Note that such a path, if it exists, is uniquely determined by q'_0, so A'' is a transducer. Furthermore, it is not hard to show that A'' accepts the graph of $g \circ f$. □

If $E \subseteq X^\omega$ is a closed rational set, then it follows from Proposition 5.4 and Proposition 5.5 that the set of all rational homeomorphisms $E \to E$ forms a group under composition. This is the **rational group** $\mathcal{R}_E$. In the case where $E = X^\omega$ and $|X| = n$, this is the **asynchronous rational group** $\mathcal{R}_n$ defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ in [GNS00]. It is proven in that paper that $\mathcal{R}_n \cong \mathcal{R}_2$ for all $n \geq 3$, and the following proposition generalizes this result. See [BBM21, Theorem 2.16] for a similar generalization.

Proposition 5.6. *If a closed rational set $E \subseteq X^\omega$ is nonempty and has no isolated points, then the corresponding rational group $\mathcal{R}_E$ is isomorphic to $\mathcal{R}_2$.*

Proof. By Proposition 5.4 and Proposition 5.5, it suffices to prove that there exists a rational homeomorphism $h\colon E \to \{0, 1\}^\omega$, since such a homeomorphism will conjugate $\mathcal{R}_E$ to $\mathcal{R}_2$. By Proposition 4.1, there exists a trim, deterministic acceptor $A = (Q, T, q_{\text{start}})$ over X^* that accepts E. We will define a transducer $A' = (Q, T', q_{\text{start}})$ over $X^* \times \{0, 1\}^*$ that determines the desired homeomorphism. The states Q and start state q_{start} are the same as for A. For the transitions, we choose for each state $q \in Q$ an ordering of the transitions from q in A:

$$q \xrightarrow{x_1} q_1, \quad q \xrightarrow{x_2} q_2, \quad \ldots, \quad q \xrightarrow{x_n} q_n.$$

We also choose a complete prefix code $\{\alpha_1, \ldots, \alpha_n\}$ of size n in $\{0, 1\}^*$, i.e. a set of n words such that each infinite binary word has exactly one of the α_i's as a prefix. (For example, if $n = 5$ then $\{0, 10, 110, 1110, 1111\}$ suffices.) Then we define the transitions from q in A' to be

$$q \xrightarrow{(x_1, \alpha_1)} q_1, \quad q \xrightarrow{(x_2, \alpha_2)} q_2, \quad \ldots, \quad q \xrightarrow{(x_n, \alpha_n)} q_n.$$

Clearly A' is a transducer. Furthermore, since the only complete prefix code that contains ε is the code $\{\varepsilon\}$ of size 1, the transducer A' will be nodegenerate as long as there does not exist a cycle in A of states that have only one outgoing transition. But any such cycle would lead to isolated points in E, so no such cycles exist and hence A' is nondegenerate. Then A' defines a rational map $h\colon E \to \{0, 1\}^\omega$, and it is not hard to see that h is surjective. □

Theorem 5.7. *Any continuous asynchronous automatic group acts on its rational boundary by rational homeomorphisms.*

Proof. Let G be a group with continuous asynchronous automatic structure (X, L). Let $x \in X$, and let $f_x \colon L \cup \partial L \to L \cup \partial L$ be the homeomorphism corresponding to the left action of x. By the definition of an asynchronous automatic group, the graph of $f_x|_L$ is a deterministic rational relation in $X^* \times X^*$. By Proposition 4.10, it follows that the graph of $f_x|_{\partial L}$ is a deterministic closed rational set $X^\omega \times X^\omega$, so $f_x|_{\partial L}$ is rational homeomorphism by Theorem 5.2. Since X generates G, this proves that G acts on ∂L by rational homeomorphisms. $\qquad\square$

Corollary 5.8. *Any continuous asynchronous automatic group embeds into the asynchronous rational group $\mathcal{R}_2$.*

Proof. Let G be a continuous asynchronous automatic group. By Theorem 2.9, free product $G * \mathbb{Z}$ has a continuous asynchronous automatic structure (X, L). Furthermore, the rational boundary ∂L has no isolated points and $G * \mathbb{Z}$ acts faithfully on ∂L. By Theorem 5.7, we know $G * \mathbb{Z}$ and acts on ∂L by rational homeomorphisms. Since the action of G on ∂L is faithful, G embeds into $\mathcal{R}_{\partial L}$, and this is isomorphic to $\mathcal{R}_2$ by Proposition 5.6. $\qquad\square$

5.2. **An example.** In order to better understand the proof of Theorem 5.2 and the related notions such as delayed automata and hedge words, we provide an explicit example.

We work with the following language

$$\{1^n 0^m \mid m, n \in \mathbb{N}\} \cup \{1^n 2^m \mid n, m \in \mathbb{N}\} \cup \{3^n 2^m \mid m, n \in \mathbb{N}\} \cup \{3^n 4^m \mid n, m \in \mathbb{N}\},$$

and we consider the nondegenerate path automaton A in Figure 6. It is easy to see that the only hedge word is '3'.

Following the strategy of the proof of Theorem 5.2, we compute the delayed automaton $A^{(1)}$ (see Figure 7).

One can see that $A^{(1)}$ is such that every Y-state is the beginning of a unique path

$$(q_0, \alpha) \xrightarrow{(\varepsilon, y_1)} (q_1, \alpha) \xrightarrow{(\varepsilon, y_2)} \cdots \xrightarrow{(\varepsilon, y_k)} (q_k, \alpha)$$

where (q_k, α) is an X-state. As a last step we retrieve the transducer pictured in Figure 8.

The associated rational map is the following:

$$f(2\psi) = 1\psi$$
$$f(0\psi) = 00\psi \qquad\qquad\qquad f(4\psi) = \psi$$
$$f(1\psi) = 1\,f(\psi) \qquad\qquad\quad f(3\psi) = h(\psi)$$

where

$$h(4\psi) = 3\psi \qquad\quad h(3\psi) = 3\,h(\psi) \qquad\quad h(2\psi) = 22\psi$$

It is easy to see that f is a representation of the map $\widetilde{f} : \mathbb{Z} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ defined as $\widetilde{f}(x, y) := (x + 1, y)$.
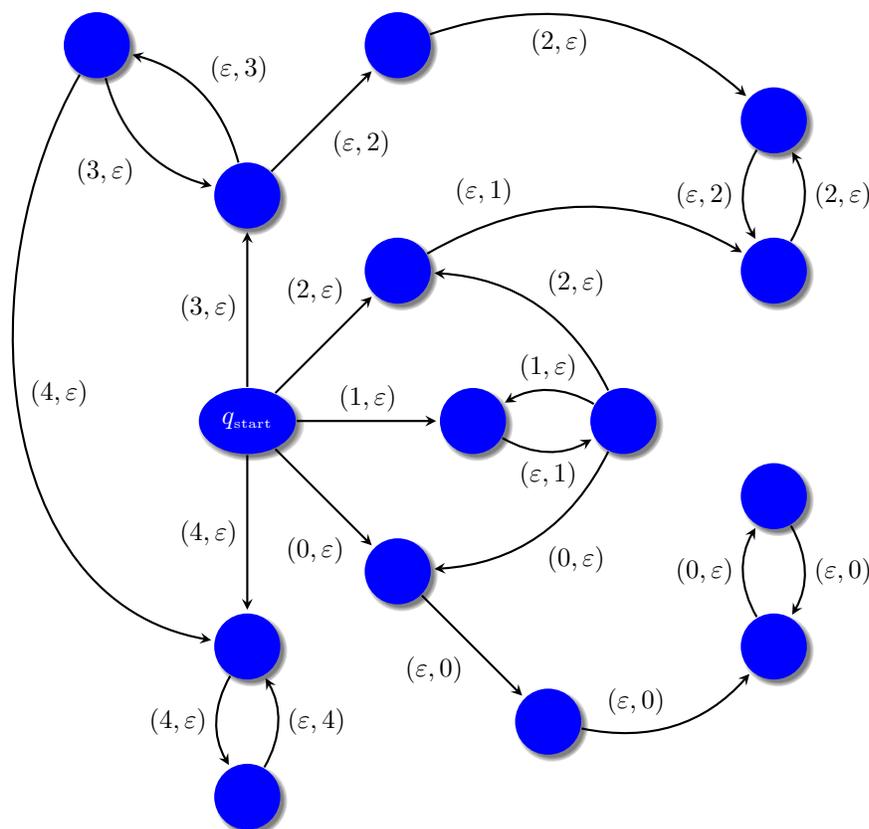
FIGURE 6. A nondegenerate path automaton A on the language
described above.

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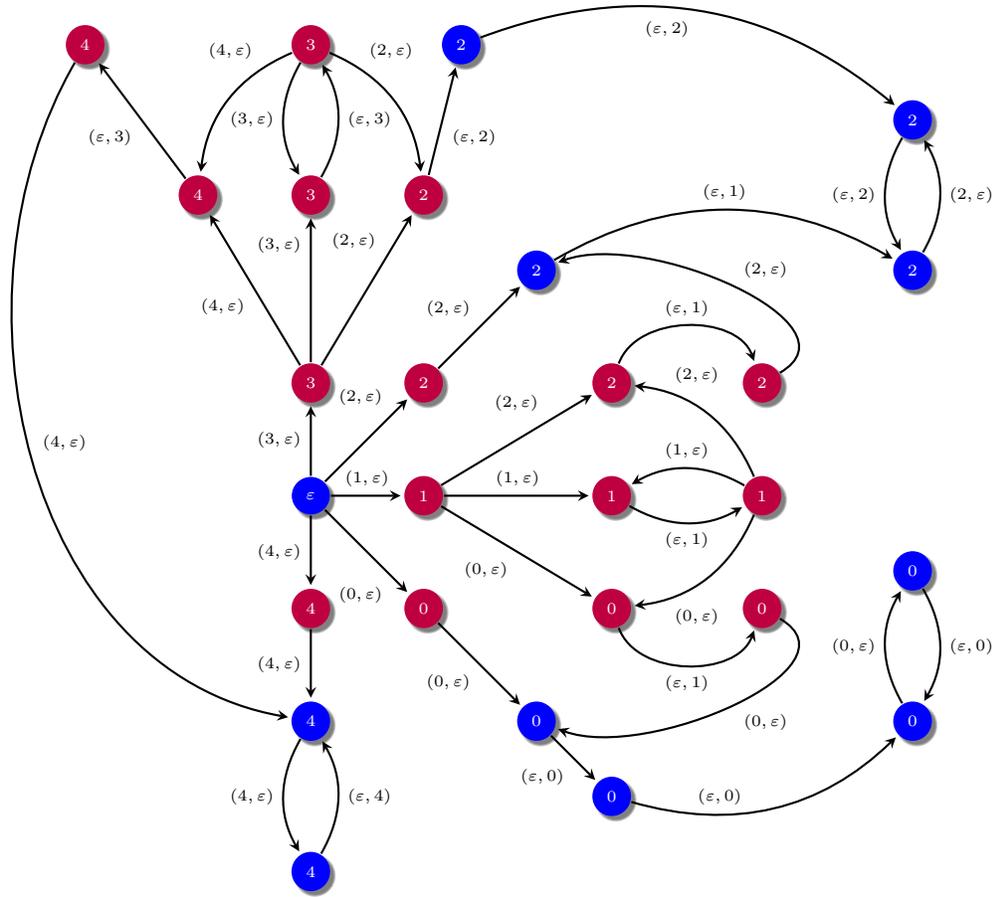   `doi:10.1007/978-1-4613-9730-4\_8`.

FIGURE 7. The delayed automaton $A^{(1)}$ with A the path automaton described in Figure 6. A state (q, α) is labeled by α. The blue states are essentially unchanged with respect to the original construction, while the red ones arise from different labelings of the original states that occupied the same position.
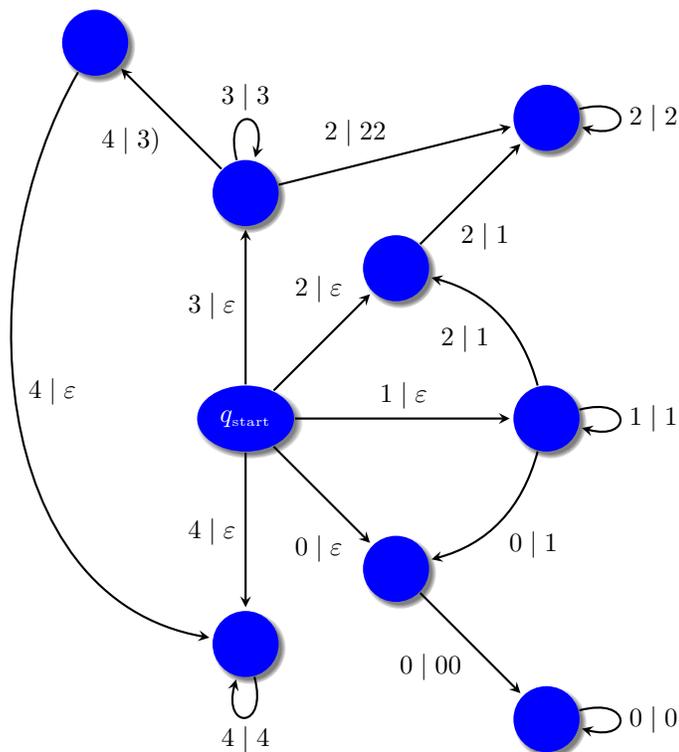
FIGURE 8. A transducer constructed from the automaton in Figure 7 and following the observation in the proof of Theorem 5.2. Again, the shape of the machine should help tracking the states in the original automaton from which the new states originate.

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