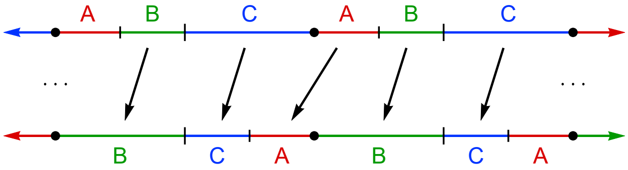
Embeddings into Finitely Presented Simple Groups



Jim Belk, University of Glasgow

Geometry & Topology Seminar, 10 October 2022

The Boone–Higman Conjecture

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

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Here a group has ***solvable word problem*** if there exists an algorithm to determine whether a given word in the generators represents the identity.

Theorem (Novikov 1955, Boone 1958)

There exist finitely presented groups with unsolvable word problem.

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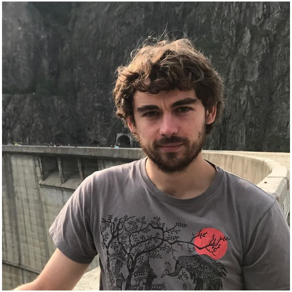
Recent progress: Many groups of interest embed into finitely presented simple groups.

Collaborators

Collin Bleak

University of St Andrews



James Hyde

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Collaborators

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SUNY University at Albany

Higman’s Embedding Theorem

Higman's Embedding Theorem

A countable group presentation

$$\langle s_1, s_2, s_3, \ldots \mid r_1, r_2, r_3, \ldots \rangle$$

is ***computable*** if there exists an algorithm that outputs the list of relations.

A group is ***computably presented*** if it admits such a presentation.

Examples

1. Any finitely presented group.
2. Any finitely generated subgroup of a finitely presented group.

Higman's Embedding Theorem (1961)

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Graham Higman, 1960

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1. *Countably generated groups with a computable presentations.*

   Follows from Higman–Neumann–Neumann 1949.

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The following groups embed into finitely presented groups:

1. *Countably generated groups with a computable presentations.*
2. *Countable abelian groups.*

   Since every such group embeds in $\bigoplus_\omega \mathbb{Q} \oplus \bigoplus_\omega \mathbb{Q}/\mathbb{Z}$.

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Problem (Higman): Find an explicit and natural example of a finitely presented group that contains $\mathbb{Q}$.

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This theorem has the form

G has a certain algorithmic property	$\Leftrightarrow$	G embeds into a certain kind of group

Question (Higman): Are there other theorems of this type?

The Boone–Higman Conjecture

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Richard J. Thompson, 2004

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Proof.

Given a presentation $\langle s_1, \ldots s_m \mid r_1, \ldots r_n \rangle$ for a simple group G and a word w, we run two simultaneous searches:

Search #1	**Search #2**
Search for a proof that $$w = 1$$ using the relations $r_1, \ldots, r_n$.	Search for a proof that $$s_1 = \cdots = s_m = 1$$ using $w = 1$ and $r_1, \ldots, r_n$.

Eventually one of the searches terminates. □

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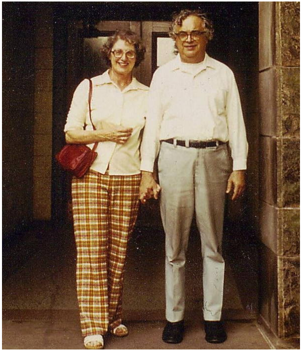
Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.



William and Eileen Boone, 1979

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Every complete theory with finitely many axioms is decidable.

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complete theory	simple group
decidable theory	decidable word problem

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Note: By a result of Clapham (1965), it would suffice to prove the conjecture for finitely presented groups.

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Like Higman's embedding theorem, this statement has the form

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As a corollary, the following groups would also embed into finitely presented simple groups:

1. Any computably presented group with solvable word problem.
2. Any countable abelian group.

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Trick: Given words $u, v \neq_G 1$, consider the group

$$G' = \left\langle G, x, t \,\middle|\, (uu^x)^t = u^x v \right\rangle.$$

G' is an HNN extension of $G * \langle x \rangle$, so G embeds into G'.

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But now v lies in the normal closure of u.

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Let

$$\sigma(G) = \left\langle G, x, t_1, t_2, \ldots \,\middle|\, (u_i u_i^x)^{t_i} = u_i^x v_i \right\rangle$$

where (u_i, v_i) is an enumeration of *all* pairs of non-identity words in G.

Theorem (Boone–Higman 1974)

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Then G embeds into $\sigma(G)$, and the normal closure of any non-identity element of G contains G.

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Then G embeds into $\sigma(G)$, and the normal closure of any non-identity element of G contains G.

The desired simple group is the union of the sequence

$$G \;\leq\; \sigma(G) \;\leq\; \sigma^2(G) \;\leq\; \sigma^3(G) \;\leq\; \cdots . \qquad \square$$

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Theorem (Thompson 1980)

Let G be a finitely generated group. Then:

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Theorem (Sacerdote 1977)

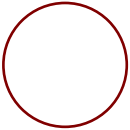
There are analogues of Boone and Higman's theorem for the order, conjugacy, power, and subgroup membership problems.

Finitely Presented Simple Groups

Thompson's Groups

In 1965, Richard J. Thompson defined three infinite groups.

F acts on the interval.



T acts on the circle.

V acts on the Cantor set.

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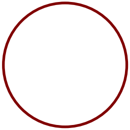

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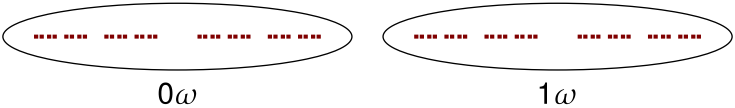

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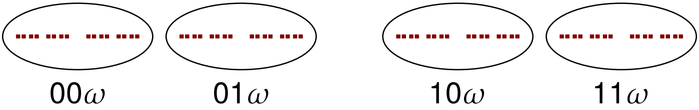

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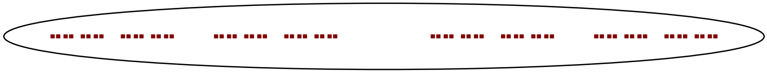

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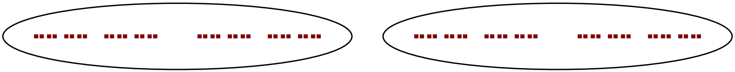

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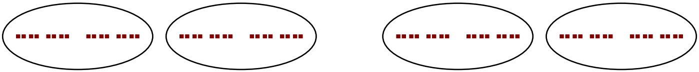

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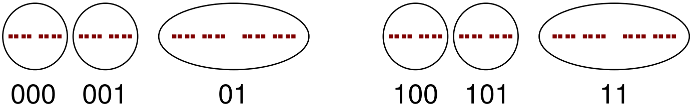

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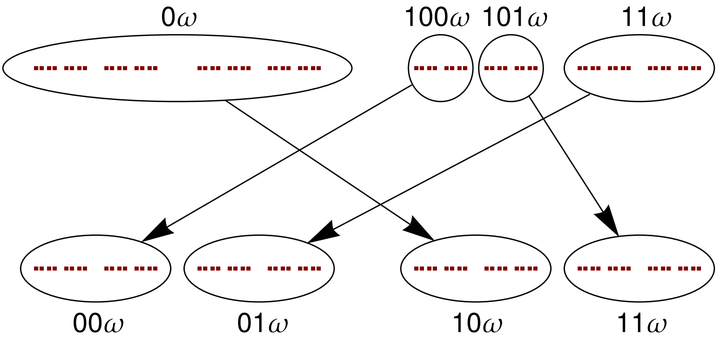
Definition of V

Thompson's group V is the group of all homeomorphisms that map "linearly" between the pieces of two dyadic subdivisions.



This group V is finitely presented and simple.

Thompson's Groups

V acts by homeomorphisms on the Cantor set.

F and *T* are subgroups of *V*.

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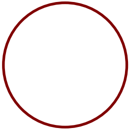

T is the subgroup of V that preserves the circular order.

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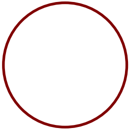

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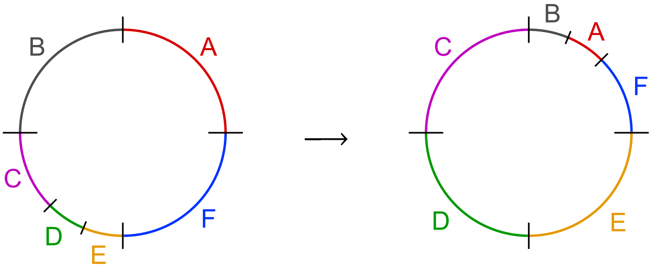
Thompson's Group T

For example, here is an element of Thompson's group T.



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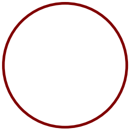

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Subgroups of V

The following groups embed into V:

1. All finite groups, free groups, and free abelian groups.
2. (Higman 1974, Brown 1987) Generalised Thompson groups F_n, T_n, and V_n.
3. (Röver 1999) Free products of finitely many finite groups.
4. (Guba–Sapir 1999, Bleak 2008) Many solvable groups.
5. (Bleak–Kassabov–Matucci 2011) $\mathbb{Q}/\mathbb{Z}$.

Grigorchuk's Group

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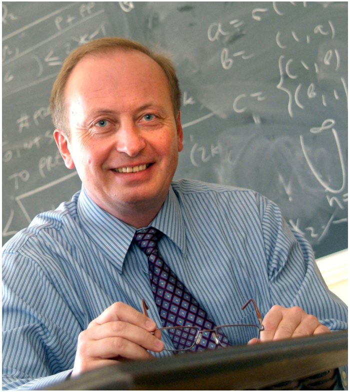
Grigorchuk (1979) defined a certain finitely generated group $\mathcal{G} \leq \mathrm{Aut}(T_2)$.



Rostislav Grigorchuk

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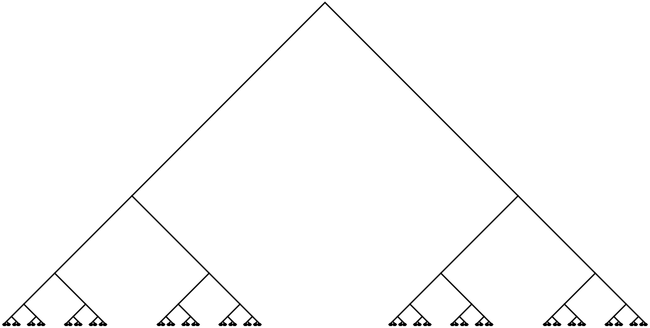

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Properties of $\mathcal{G}$ (Grigorchuk 1979 and 1984)

- $\mathcal{G}$ is a solution to the Burnside problem: it is infinite and finitely generated, and every element has finite order.
- $\mathcal{G}$ has intermediate growth: the number of elements of length less than n grows like $\exp\left(n^{0.7675}\right)$ (Erschler–Zheng 2020).

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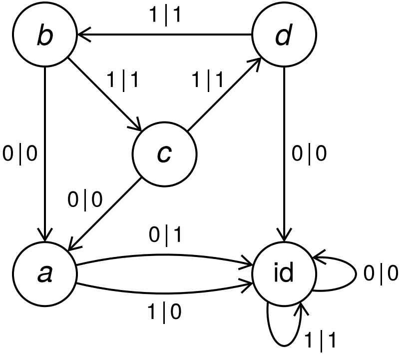
The action of $\mathcal{G}$ on binary sequences in $\{0, 1\}^\omega$ can be described by ***automata***.



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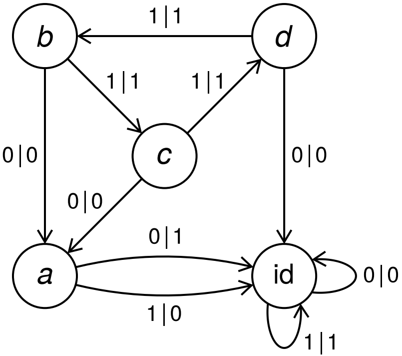

$c(1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \cdots)$

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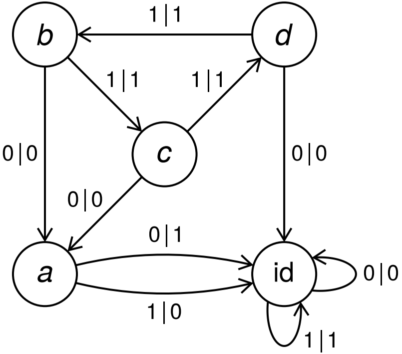

$$c(1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \cdots)$$
$$= 1 \cdot d(1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \cdots)$$

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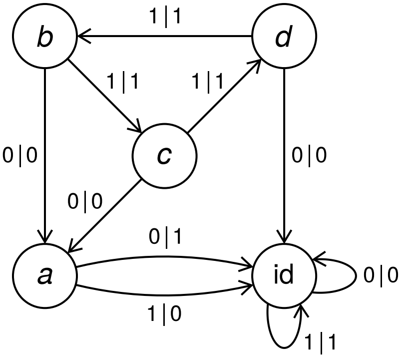

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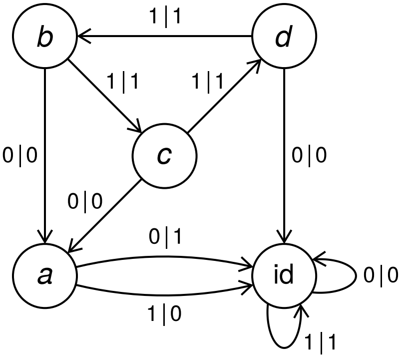

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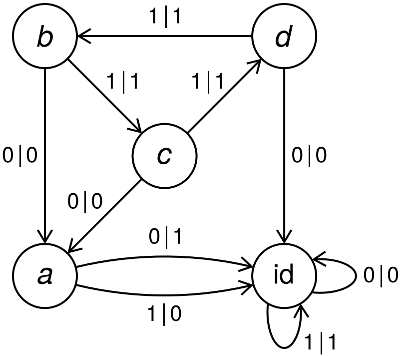

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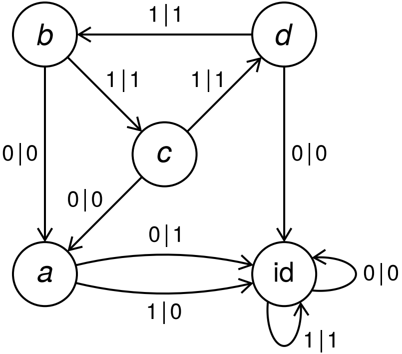

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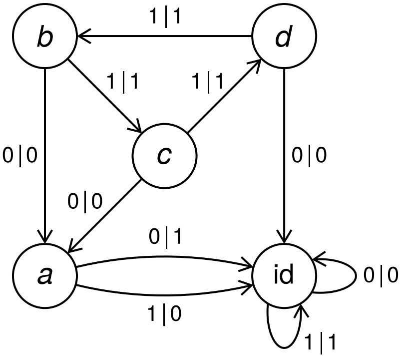

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Every element of $\mathcal{G}$ has such an automaton.

Grigorchuk's Group

Does $\mathcal{G}$ embed into Thompson's group V?

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Theorem (Röver 1999)

Yes. The group $V\mathcal{G}$ generated by V and $\mathcal{G}$ is finitely presented and simple!

Nekrashevych Groups

Nekrashevych (2005) generalized Grigorchuk's group to the family of ***contracting self-similar groups*** $G \leq \mathrm{Aut}(T_d)$.

Volodymyr Nekrashevych

Nekrashevych Groups

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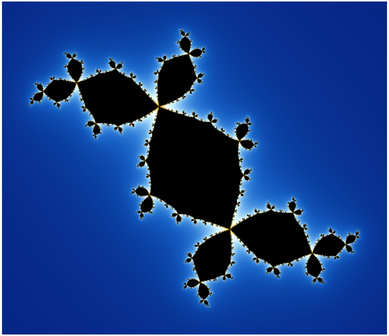
These include the "iterated monodromy groups" associated to complex dynamical systems.



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Nekrashevych also gave necessary and sufficient conditions for $V_d G$ to be simple.

This gives Boone–Higman embeddings for *some* contracting self-similar groups.

Brin’s Groups

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Matthew Brin

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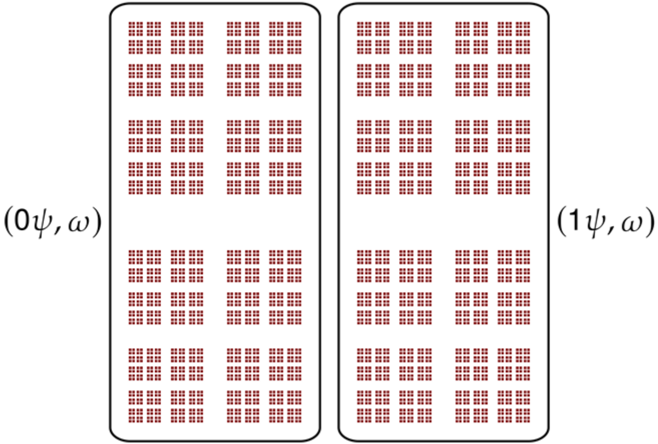

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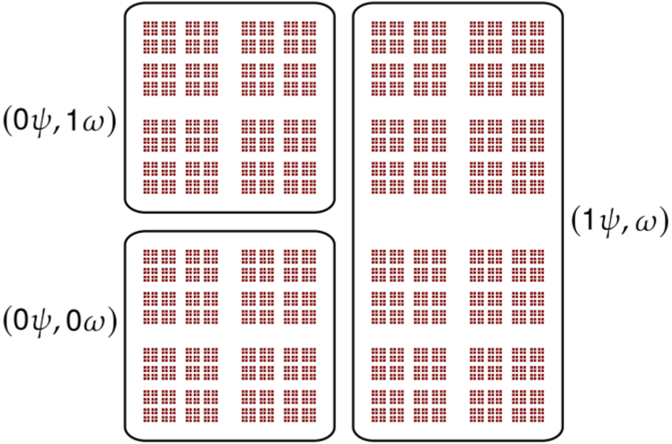

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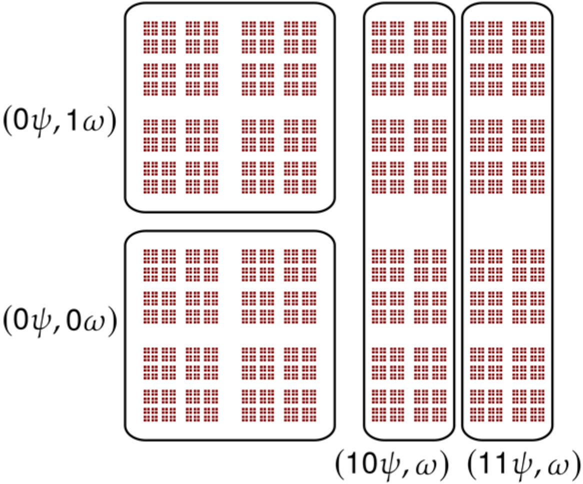
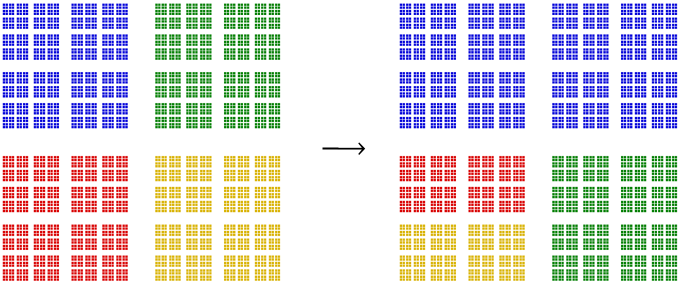


Brin's Groups

Elements of $2V$ map "linearly" between two subdivisions.



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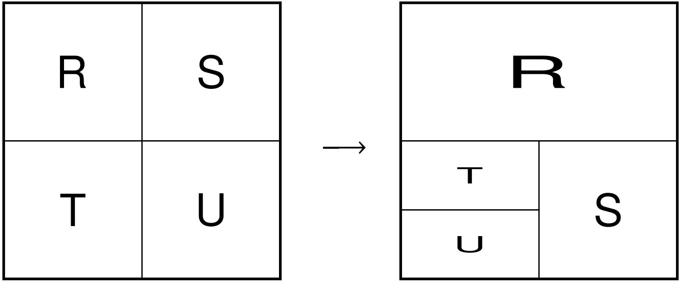

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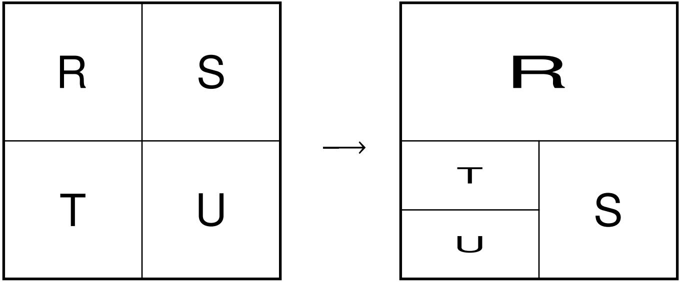


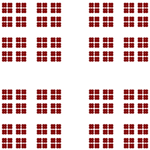
Theorem (Brin 2004)

The group $2V$ is finitely presented and simple.

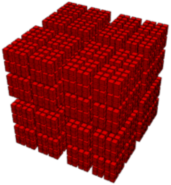
Brin's Groups

Brin defined a family of groups nV ($n \geq 1$) similarly, with $1V = V$.

V



$2V$



$3V$

$\ldots$

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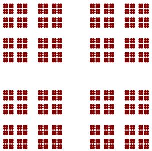

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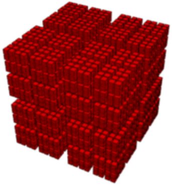

$3V$

$\cdots$

Theorem (Brin 2009)

The group nV is finitely presented and simple for all $n \geq 1$.

Brin's Groups

These groups have very interesting algorithmic properties.

Theorem (B–Bleak 2014)

The order problem in nV is unsolvable for $n \geq 2$

Theorem (B–Bleak–Matucci 2016)

The subgroup membership problem in nV is unsolvable for $n \geq 2$.

Theorem (Salo 2020)

The conjugacy problem in nV is unsolvable for $n \geq 2$.

Right-angled Artin groups

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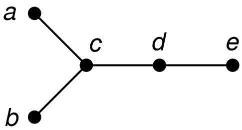
Given a finite graph Γ



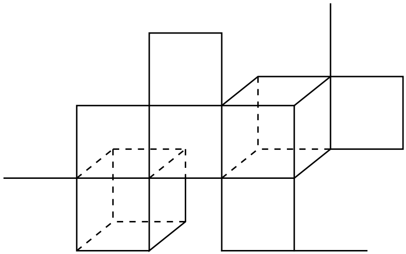
the corresponding **right-angled Artin group (RAAG)** has one generator for each vertex, with edges corresponding to generators that commute:

$$A_\Gamma = \langle a, b, c, d, e \mid ac = ca,\ bc = cb,\ cd = dc,\ de = ed \rangle.$$

Right-angled Artin groups

RAAG's are very interesting from an embeddings perspective.

Haglund and Wise (2008) have shown that the fundamental group of any (compact) ***special cube complex*** embeds into a RAAG.



Such complexes were crucial to the proof of the virtual Haken conjecture (Agol 2012).

Virtually Special Groups

A group is ***virtually special*** if it has a finite-index subgroup isomorphic to $\pi_1(X)$ for some special cube complex X.

The class of virtually special groups includes:

1. All right-angled Artin groups.
2. (Haglund–Wise 2010) All finitely generated Coxeter groups.
3. (Agol 2012) All cubulated hyperbolic groups.
4. (Agol 2012) Fundamental groups of finite-volume hyperbolic 3-manifolds.
5. (Przytycki–Wise 2012) Fundamental groups of compact Riemannian 3-manifolds of non-positive curvature.

Virtually Special Groups

Theorem (B–Bleak–Matucci 2016)

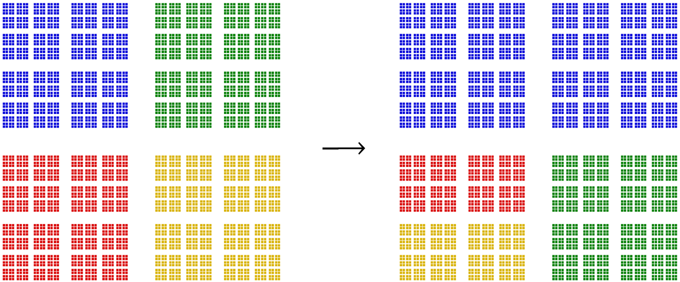
If a group G has a finite-index subgroup that embeds into a RAAG, then G embeds into one of Brin's groups nV.

Corollary

Every virtually special group embeds into a finitely presented simple group.

Virtually Special Groups

Salo (2021) has recently shown that in fact all virtually special groups embed into Brin's group $2V$.



Countable Abelian Groups

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Problem (Higman): Find an explicit and natural example of a finitely presented group that contains $\mathbb{Q}$.

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In 1999, Martin Bridson and Pierre de la Harpe submitted this question to the Kourovka notebook as a “well-known” problem.

Bridson also mentioned this problem in his article on Geometric and Combinatorial Group Theory in the *Princeton Companion to Mathematics* (2008).

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In 2020, James Hyde, Francesco Matucci, and I noticed an elementary solution.

Countable Abelian Groups

Recall that Thompson's group T acts on S^1.

A ***lift*** of an element $g \in T$ is a homeomorphism $\overline{g} \colon \mathbb{R} \to \mathbb{R}$ that makes the following diagram commute:

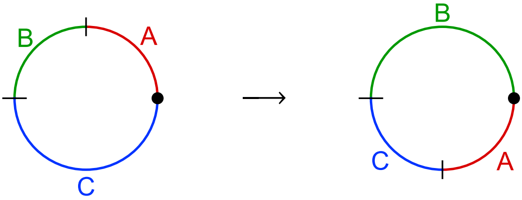
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\overline{g}} & \mathbb{R} \\ \big\downarrow & & \big\downarrow \\ S^1 & \xrightarrow[g]{} & S^1 \end{array}$$

Note: If $\overline{g}$ is a lift of g then so is $\overline{g} + n$ for any $n \in \mathbb{Z}$.

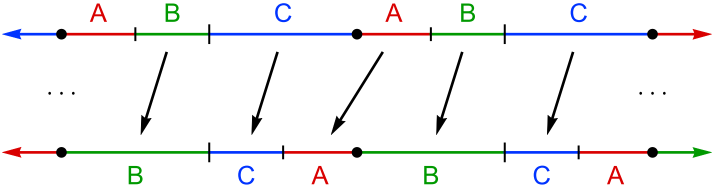
Let $\overline{T}$ be the group of all lifts of elements of T.

Countable Abelian Groups

For example, here's an element of T:



and here's one possible lift in $\overline{T}$:



Countable Abelian Groups

Theorem (B–Hyde–Matucci 2020)
The group $\overline{T}$ is finitely presented and contains $\mathbb{Q}$.

$$\overline{T} = \left\langle a, b \;\middle|\; a^4 b^{-3}, (ba)^5 b^{-9}, [bab, a^2 baba^2], [bab, a^2 b^2 a^2 baba^2 ba^2] \right\rangle$$

Note: We did not introduce this group $\overline{T}$. It had previously appeared in the work of Ghys and Sergiescu (1987).

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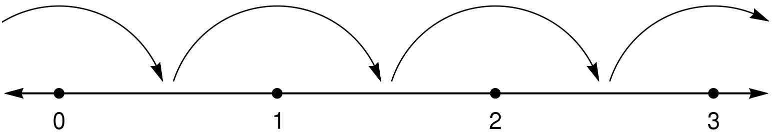

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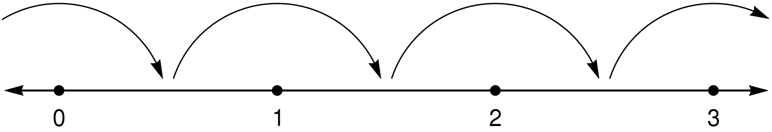
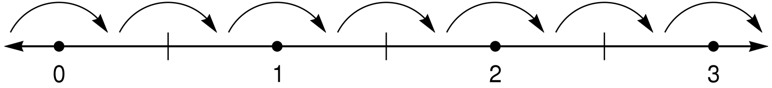


It's easy to find a square root f_2 of f_1:



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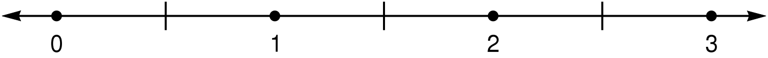

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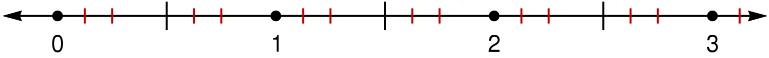

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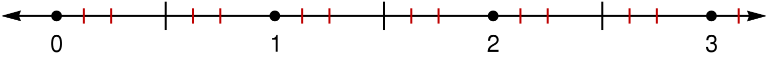
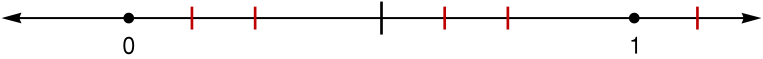


Next, construct a fourth root f_4 of f_3:



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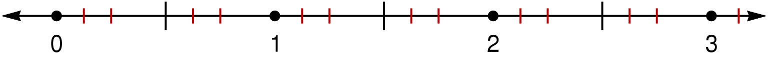

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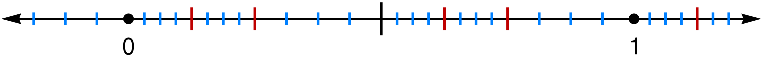

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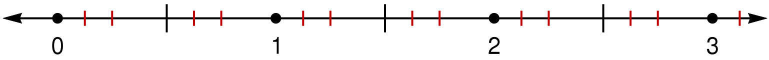

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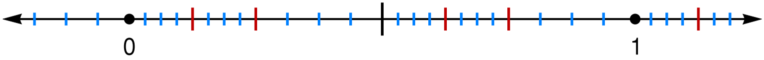

Then $\langle f_1, f_2, f_3, f_4, \ldots \rangle \cong \mathbb{Q}$. □

Countable Abelian Groups

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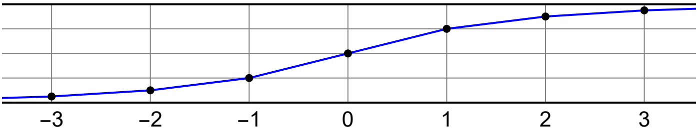

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We prove that the group $V\overline{T}$ generated by V and $\overline{T}$ is finitely presented, simple, and contains $\bigoplus_\omega \mathbb{Q} \oplus \bigoplus_\omega \mathbb{Q}/\mathbb{Z}$. □

Twisting Brin’s Groups

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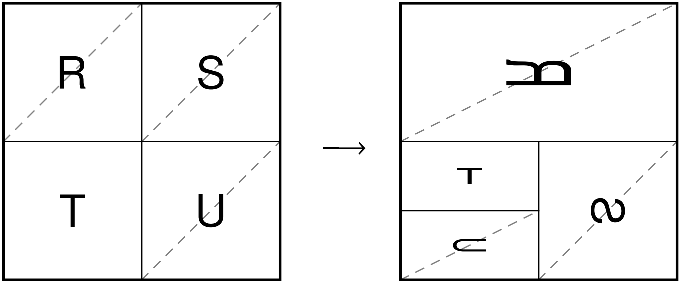
In 2020, Matthew Zaremsky and I considered a "twisted" version of Brin's group $2V$.



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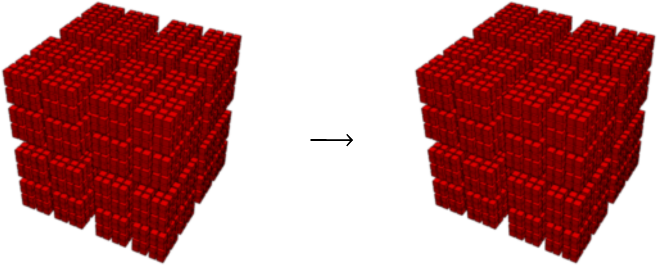
In general, you can twist nV by any group of permutations of $\{1, \ldots, n\}$.



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Corollary (B–Zaremsky 2020)

Any finitely generated group G embeds isometrically into a finitely generated simple group.

Twisting Brin's Groups

We can also get finitely presented simple groups.

Theorem (B–Zaremsky 2020, Zaremsky 2022)

Suppose:

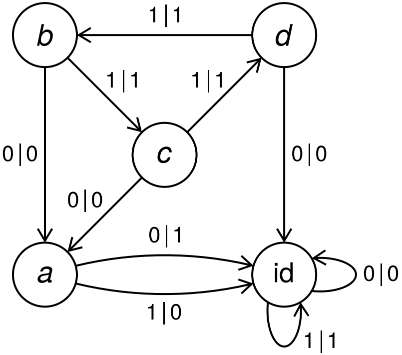
1. *G is finitely presented,*
2. *G acts highly transitively on a set X, and*
3. *Stabilizers of finite subsets of X are finitely generated.*

Then the resulting twisted ωV is a finitely presented simple group that contains G.

Twisting Brin's Groups

Corollary

Every contracting self-similar embeds into a finitely presented simple group.



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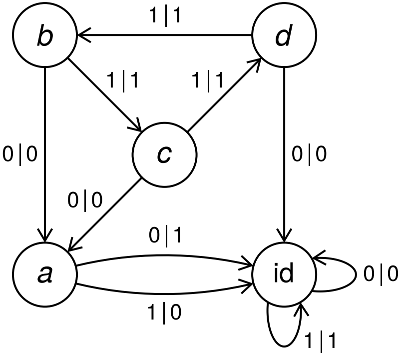

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The Nekrashevych group $V_d G$ is finitely presented, highly transitive on any orbit, and has finitely generated stabilizers, so the resulting twisted ωV is finitely presented and simple.

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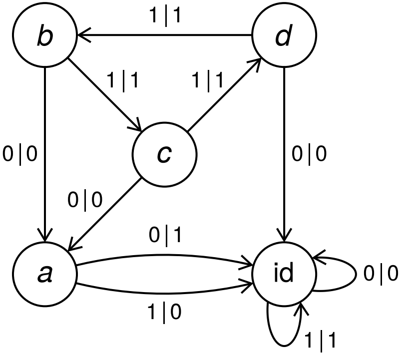

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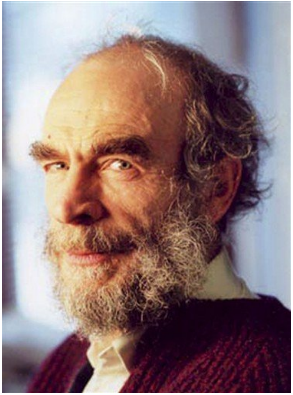
We can similarly handle many other "Thompson-like" groups.

Hyperbolic Groups

Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.



Misha Gromov

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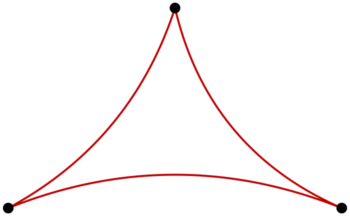

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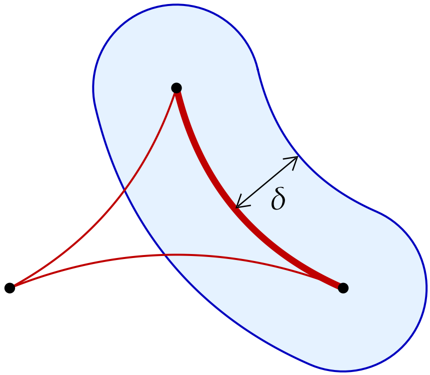

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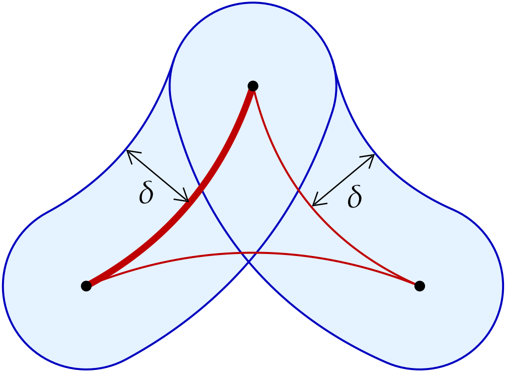

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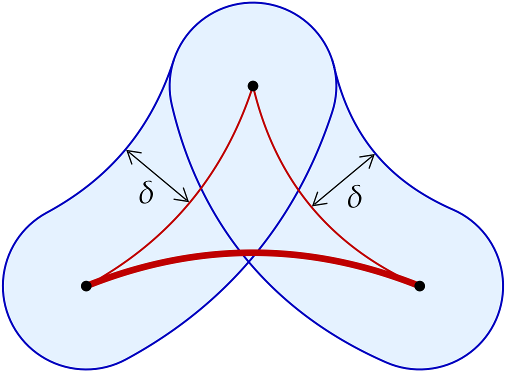

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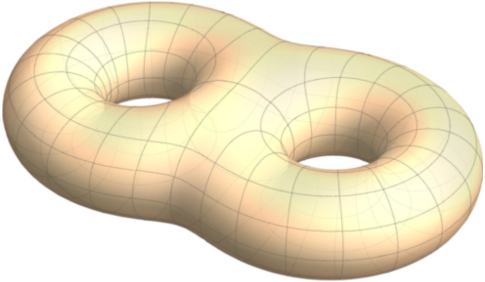
For example, the fundamental group of any compact hyperbolic manifold is a hyperbolic group.



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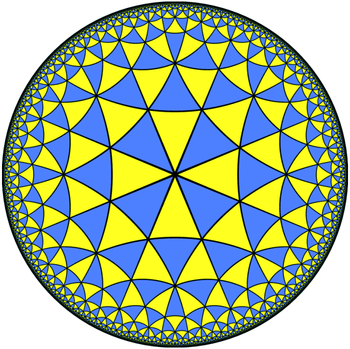
In a certain precise sense, almost every finitely presented group is hyperbolic (Ol'Shanskii 1991).

Hyperbolic Groups

Theorem (B–Bleak–Matucci–Zaremsky 2022)

Every hyperbolic group G embeds into a finitely presented simple group.



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4. Conclude that $V[G]$ embeds into a twisted ωV which is finitely presented and simple.

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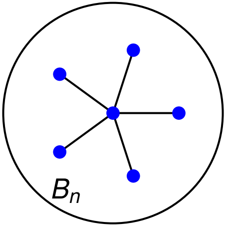

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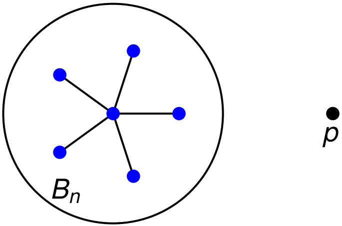

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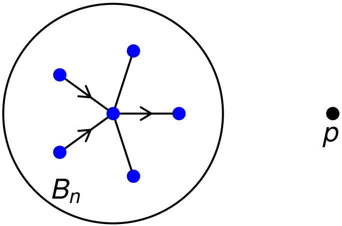

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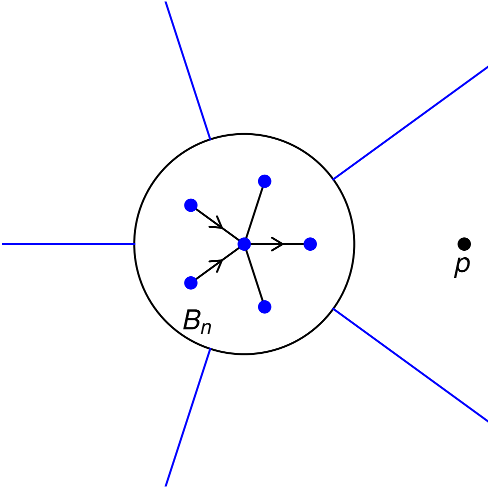

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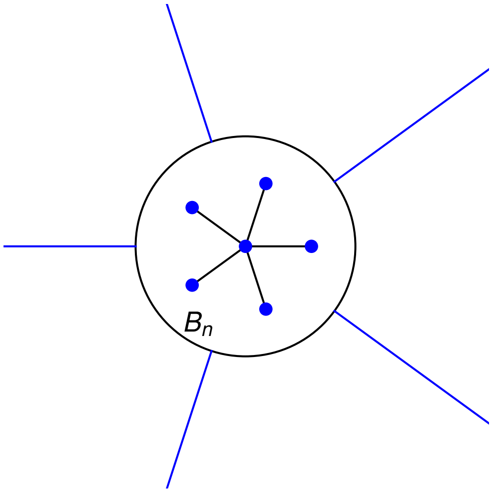

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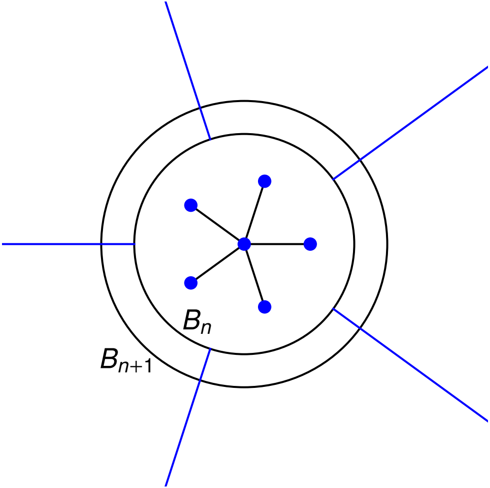

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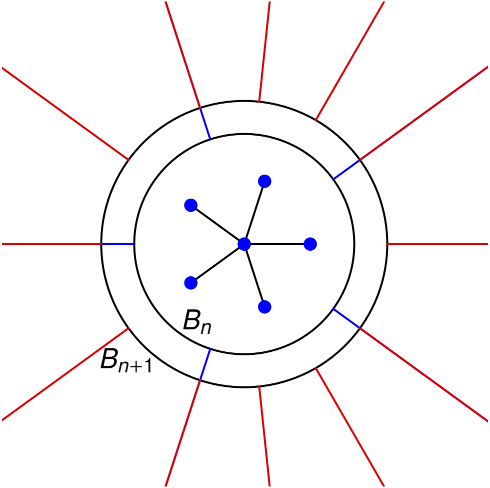

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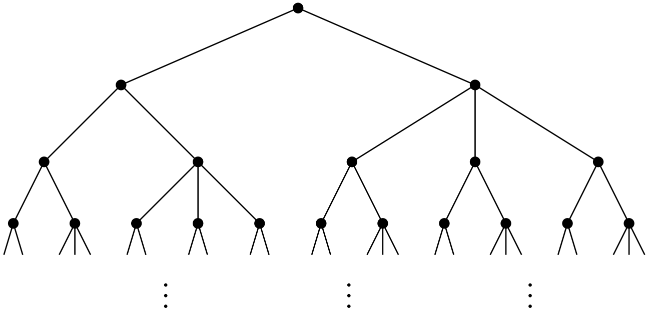

This is the ***tree of atoms***. Its space of ends is $\partial_h G$.

Hyperbolic Groups

Theorem (B–Bleak–Matucci 2018)

If G is a hyperbolic group, then:

1. *The tree of atoms has a self-similar structure, and*
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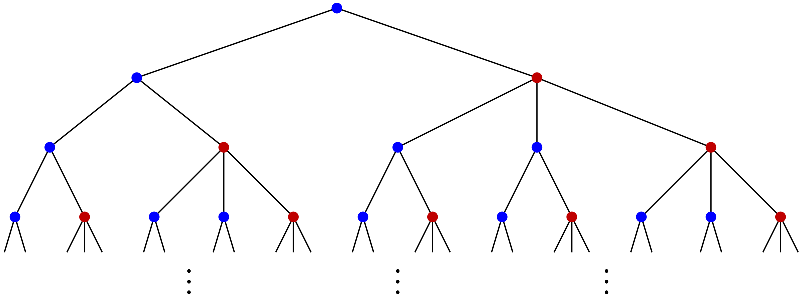

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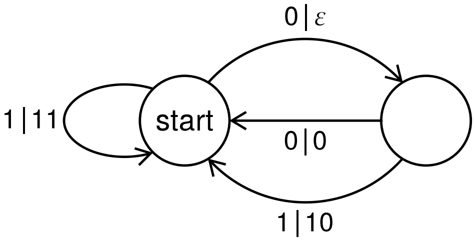

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The action of G on $\partial_h G$ is contracting, and hence $V[G]$ is finitely presented.

Mapping Class Groups

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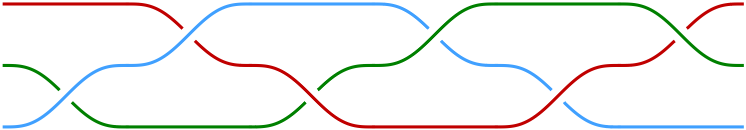
Open Question: Do mapping class groups and braid groups embed into finitely presented simple groups?



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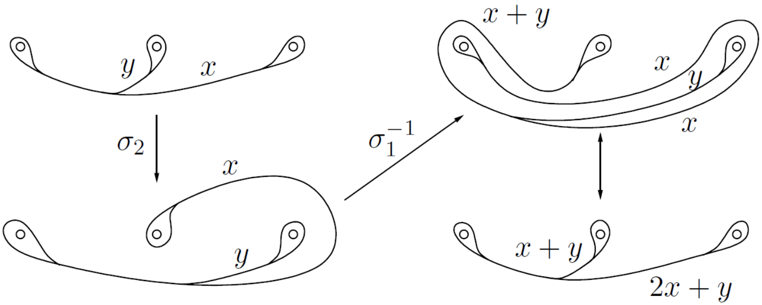

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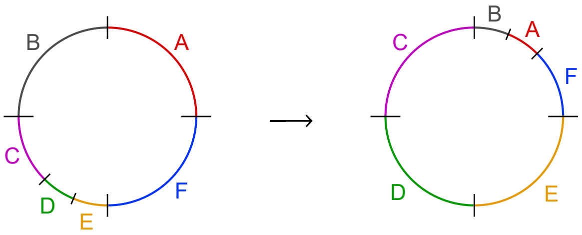

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If PIP(S^n) is finitely presented and simple, this would give Boone–Higman embeddings for all mapping class groups.

The End