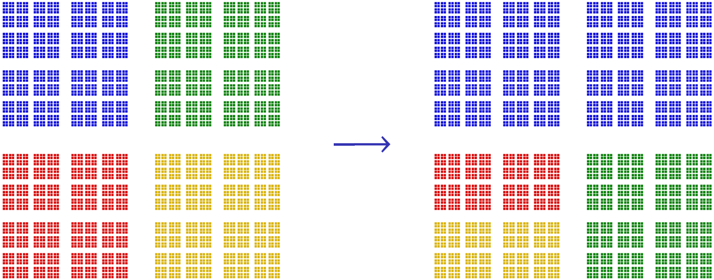
Hyperbolic Groups Satisfy the Boone–Higman Conjecture



Jim Belk, University of Glasgow

Collaborators

Collin Bleak

U. of St Andrews

Francesco Matucci

U. of Milano–Bicocca

Matthew Zaremsky

SUNY at Albany

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

Note: The $(\Leftarrow)$ direction is easy, but $(\Rightarrow)$ is open.

The Boone–Higman Conjecture

Higman's Embedding Theorem

Let G be a f.g. group, and let R be the set of all words for the identity.

1. G is **computably presented** if R is computably enumerable.
2. G has **solvable word problem** if R is computable.

Higman's Embedding Theorem

Let G be a f.g. group, and let R be the set of all words for the identity.

1. G is **computably presented** if R is computably enumerable.
2. G has **solvable word problem** if R is computable.

Note: G is computably presented if and only if G has a presentation

$$\langle S \mid r_1, r_2, r_3, \ldots \rangle$$

whose relations $r_1, r_2, r_3, \ldots$ are computably enumerable.

Higman's Embedding Theorem

Let G be a f.g. group, and let R be the set of all words for the identity.

1. G is **computably presented** if R is computably enumerable.
2. G has **solvable word problem** if R is computable.

Note: G is computably presented if and only if G has a presentation

$$\langle S \mid r_1, r_2, r_3, \ldots \rangle$$

whose relations $r_1, r_2, r_3, \ldots$ are computably enumerable.

Note 2: Any f.g. subgroup of a finitely presented group is computably presented.

Higman's Embedding Theorem (1961)

Let G be a f.g. group. Then:

G is computably presented	$\Leftrightarrow$	*G embeds into a finitely presented group*

Graham Higman, 1960

Higman's Embedding Theorem (1961)

Let G be a f.g. group. Then:

G is computably presented	$\Leftrightarrow$	*G embeds into a finitely presented group*

Higman's Embedding Theorem (1961)

Let G be a f.g. group. Then:

G *is computably presented*	$\Leftrightarrow$	G *embeds into a finitely presented group*

This theorem has the form

G has a certain algorithmic property	$\Leftrightarrow$	G embeds into a certain kind of group

Question (Higman): Are there other theorems of this type?

Higman's Embedding Theorem (1961)

Let G be a f.g. group. Then:

G is computably presented	$\Leftrightarrow$	*G embeds into a finitely presented group*

This theorem has the form

G has a certain algorithmic property	$\Leftrightarrow$	G embeds into a certain kind of group

Question (Higman): Are there other theorems of this type?

For example, is there a version for groups with solvable word problem?

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Richard J. Thompson, 2004

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Proof.

Given a presentation $\langle s_1, \ldots s_m \mid r_1, \ldots r_n \rangle$ for a simple group G and a word w, we run two simultaneous searches:

Search #1

Search for a proof that

$$w = 1$$

using the relations $r_1, \ldots, r_n$.

Search #2

Search for a proof that

$$s_1 = \cdots = s_m = 1$$

using $w = 1$ and $r_1, \ldots, r_n$.

Eventually one of the searches terminates. □

An Observation

Observation (Kuznecov 1958, Thompson 1969)

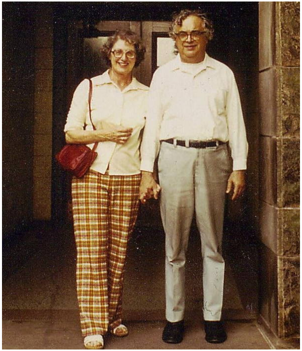
Every finitely presented simple group has solvable word problem.

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.



William and
Eileen Boone, 1979

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.

Theorem (Boone–Higman 1974)

Let G be a finitely generated group. Then:

$$\boxed{\textit{G has solvable word problem}} \iff \boxed{\textit{G embeds into a simple subgroup of a finitely presented group}}$$

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.

Theorem (Thompson 1980)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a f.g. simple subgroup of a finitely presented group*

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G *has solvable word problem*	$\Leftrightarrow$	G *embeds into a finitely presented simple group*

Progress so far

The following groups embed into finitely presented simple groups:

1. Subgroups of V, e.g. free groups, free abelian groups, etc.
2. (Scott 1984) $\mathrm{GL}_n(\mathbb{Z})$ for all $n \geq 2$.
3. (Röver 1999) Grigorchuk's group.
4. (Hsu–Wise 1999) Finitely generated right-angled Artin groups.
   (Haglund–Wise 2010) Finitely generated Coxeter groups.
   (Agol 2012) Cubulated hyperbolic groups.
5. (B–Hyde–Matucci 2023) All countable abelian groups.
6. (BBMZ 2023) All hyperbolic groups.
   (BBMZ 2023) All contracting self-similar groups.

Progress so far

The following groups embed into finitely presented simple groups:

1. Subgroups of V, e.g. free groups, free abelian groups, etc.
2. (Scott 1984) $\mathrm{GL}_n(\mathbb{Z})$ for all $n \geq 2$.
3. (Röver 1999) Grigorchuk's group.
4. (Hsu–Wise 1999) Finitely generated right-angled Artin groups.
   (Haglund–Wise 2010) Finitely generated Coxeter groups.
   (Agol 2012) Cubulated hyperbolic groups.
5. (B–Hyde–Matucci 2023) All countable abelian groups.
6. (BBMZ 2023) All hyperbolic groups.
   (BBMZ 2023) All contracting self-similar groups.

Progress so far

The following groups embed into finitely presented simple groups:

1. Subgroups of V, e.g. free groups, free abelian groups, etc.
2. (Scott 1984) $\mathrm{GL}_n(\mathbb{Z})$ for all $n \geq 2$.
3. (Röver 1999) Grigorchuk's group.
4. (Hsu–Wise 1999) Finitely generated right-angled Artin groups.
   (Haglund–Wise 2010) Finitely generated Coxeter groups.
   (Agol 2012) Cubulated hyperbolic groups.
5. (B–Hyde–Matucci 2023) All countable abelian groups.
6. (BBMZ 2023) All hyperbolic groups.
   (BBMZ 2023) All contracting self-similar groups.

Progress so far

The following groups embed into finitely presented simple groups:

1. Subgroups of V, e.g. free groups, free abelian groups, etc.
2. (Scott 1984) $\mathrm{GL}_n(\mathbb{Z})$ for all $n \geq 2$.
3. (Röver 1999) Grigorchuk's group.
4. (Hsu–Wise 1999) Finitely generated right-angled Artin groups.
   (Haglund–Wise 2010) Finitely generated Coxeter groups.
   (Agol 2012) Cubulated hyperbolic groups.
5. (B–Hyde–Matucci 2023) All countable abelian groups.
6. (BBMZ 2023) All hyperbolic groups.
   (BBMZ 2023) All contracting self-similar groups.

Progress so far

The following groups embed into finitely presented simple groups:

1. Subgroups of V, e.g. free groups, free abelian groups, etc.
2. (Scott 1984) $\mathrm{GL}_n(\mathbb{Z})$ for all $n \geq 2$.
3. (Röver 1999) Grigorchuk's group.
4. (Hsu–Wise 1999) Finitely generated right-angled Artin groups.
   (Haglund–Wise 2010) Finitely generated Coxeter groups.
   (Agol 2012) Cubulated hyperbolic groups.
5. (B–Hyde–Matucci 2023) All countable abelian groups.
6. (BBMZ 2023) All hyperbolic groups.
   (BBMZ 2023) All contracting self-similar groups.

Progress so far

The following groups embed into finitely presented simple groups:

1. Subgroups of V, e.g. free groups, free abelian groups, etc.
2. (Scott 1984) $\mathrm{GL}_n(\mathbb{Z})$ for all $n \geq 2$.
3. (Röver 1999) Grigorchuk's group.
4. (Hsu–Wise 1999) Finitely generated right-angled Artin groups.
   (Haglund–Wise 2010) Finitely generated Coxeter groups.
   (Agol 2012) Cubulated hyperbolic groups.
5. (B–Hyde–Matucci 2023) All countable abelian groups.
6. (BBMZ 2023) All hyperbolic groups.
   (BBMZ 2023) All contracting self-similar groups.

Progress so far

The following groups embed into finitely presented simple groups:

1. Subgroups of V, e.g. free groups, free abelian groups, etc.
2. (Scott 1984) $\mathrm{GL}_n(\mathbb{Z})$ for all $n \geq 2$.
3. (Röver 1999) Grigorchuk's group.
4. (Hsu–Wise 1999) Finitely generated right-angled Artin groups.
   (Haglund–Wise 2010) Finitely generated Coxeter groups.
   (Agol 2012) Cubulated hyperbolic groups.
5. (B–Hyde–Matucci 2023) All countable abelian groups.
6. (BBMZ 2023) All hyperbolic groups.
   (BBMZ 2023) All contracting self-similar groups.

Open Questions

Which of the following groups embed into finitely presented simple groups?

1. Braid groups, mapping class groups, $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$.
2. (Non-solvable) Baumslag-Solitar groups $BS(m, n)$.
3. One-relator groups (without torsion).
4. $\mathrm{GL}_n(\mathbb{Q})$.
5. Finitely generated metabelian groups.
6. Free by cyclic groups.
7. $\mathrm{CAT}(0)$ groups.

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

Note: The $(\Leftarrow)$ direction is easy, but $(\Rightarrow)$ is open.

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Sketch of Proof.

1. Embed every hyperbolic group into a “Thompson-like” group.
2. Embed these “Thompson-like” groups into finitely presented simple groups. □

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Sketch of Proof.

1. Embed every hyperbolic group into a "Thompson-like" group.
2. Embed these "Thompson-like" groups into finitely presented simple groups. □

Note: Our "Thompson-like" groups belong to a new class, which we call **rational similarity groups (RSGs)**. Specifically, they are full, contracting RSGs.

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Sketch of Proof.

1. Embed every hyperbolic group into a “Thompson-like” group.
2. Embed these “Thompson-like” groups into finitely presented simple groups. □

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Sketch of Proof.

1. Embed every hyperbolic group into a "Thompson-like" group.
2. Embed these "Thompson-like" groups into finitely presented simple groups. □

Right now, let's talk about step #2.

Boone–Higman embeddings of Thompson-like groups

Embeddings of “Thompson-like” groups

“Thompson-like” groups aren’t always simple, e.g. $V_{n,r}$ is not simple if n is odd.

(B–Zaremsky 2022) introduced some robust technology for embedding such groups into finitely presented simple groups.

Embeddings of "Thompson-like" groups

"Thompson-like" groups aren't always simple, e.g. $V_{n,r}$ is not simple if n is odd.

(B–Zaremsky 2022) introduced some robust technology for embedding such groups into finitely presented simple groups.

Theorem (Zaremsky 2022)

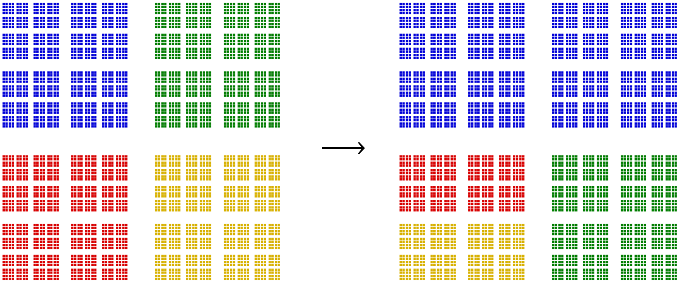
Let G be a group acting faithfully on a countable set X. Suppose:

1. *G is finitely presented,*
2. *The stabilizer of any finite subset of X is finitely generated, and*
3. *G is oligomorphic, i.e. for each n there are finitely many orbits of n-element subsets of X.*

Then G embeds into a finitely presented simple group.

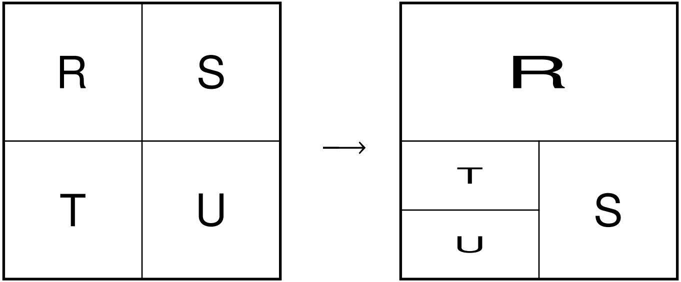
How is this accomplished?

Recall that the **Brin–Thompson group 2*V*** acts on the Cantor square $C \times C$.



How is this accomplished?

Recall that the **Brin–Thompson group 2*V*** acts on the Cantor square $C \times C$.



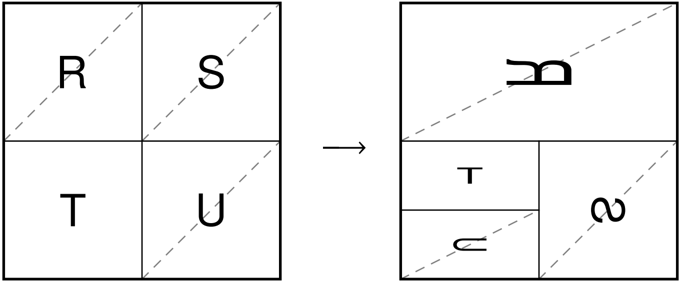
How is this accomplished?

Recall that the **Brin–Thompson group $2V$** acts on the Cantor square $C \times C$.

How is this accomplished?

Recall that the **Brin–Thompson group 2V** acts on the Cantor square $C \times C$.

Twisted 2V is similar, but rectangles are allowed to flip diagonally.



How is this accomplished?

Recall that the **Brin–Thompson group 2*V*** acts on the Cantor square $C \times C$.

Twisted 2*V* is similar, but rectangles are allowed to flip diagonally.

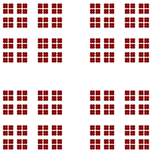
How is this accomplished?

Recall that the **Brin–Thompson group 2V** acts on the Cantor square $C \times C$.

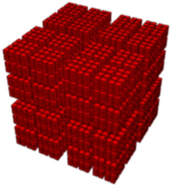
Twisted 2V is similar, but rectangles are allowed to flip diagonally.

More generally, nV acts on a Cantor n-cube C^n, and we can twist by any group of permutations of $\{1, \ldots, n\}$.

V



$2V$



$3V$

$\cdots$

How is this accomplished?

Recall that the **Brin–Thompson group 2V** acts on the Cantor square $C \times C$.

Twisted 2V is similar, but rectangles are allowed to flip diagonally.

More generally, nV acts on a Cantor n-cube C^n, and we can twist by any group of permutations of $\{1, \ldots, n\}$.

How is this accomplished?

Recall that the **Brin–Thompson group 2V** acts on the Cantor square $C \times C$.

Twisted 2V is similar, but rectangles are allowed to flip diagonally.

More generally, nV acts on a Cantor n-cube C^n, and we can twist by any group of permutations of $\{1, \ldots, n\}$.

You can even twist ωV by a group of permutations of an infinite set to get a **twisted ωV**.

How is this accomplished?

Recall that the **Brin–Thompson group 2V** acts on the Cantor square $C \times C$.

Twisted 2V is similar, but rectangles are allowed to flip diagonally.

More generally, nV acts on a Cantor n-cube C^n, and we can twist by any group of permutations of $\{1, \ldots, n\}$.

You can even twist ωV by a group of permutations of an infinite set to get a **twisted ωV**.

If we twist ωV by G, then G embeds into the resulting twisted ωV. Under the right circumstances, this twisted ωV is finitely presented and simple.

Application to self-similar groups

Theorem (Zaremsky 2022)

Let G be a group acting faithfully on a countable set X. Suppose:

1. *G is finitely presented,*
2. *The stabilizer of any finite subset of X is finitely generated, and*
3. *G is oligomorphic.*

Then G embeds into a finitely presented simple group.

Application to self-similar groups

Theorem (Zaremsky 2022)

Let G be a group acting faithfully on a countable set X. Suppose:

1. *G is finitely presented,*
2. *The stabilizer of any finite subset of X is finitely generated, and*
3. *G is oligomorphic.*

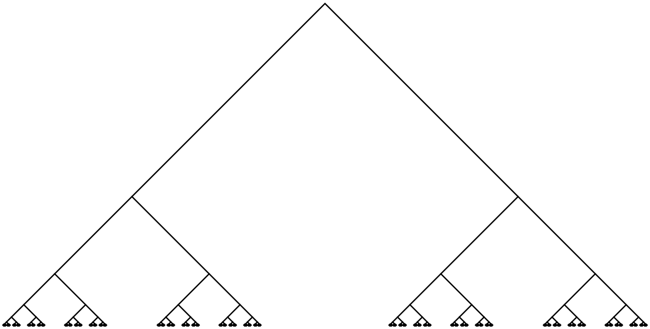
Then G embeds into a finitely presented simple group.

Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every contracting self-similar group embeds into a finitely presented simple group.

Application to self-similar groups

A **self-similar group** G (e.g. Grigorchuk's group) is a group of automorphisms of the infinite, rooted binary tree $\mathcal{T}_d$ that is closed under restrictions.



Application to self-similar groups

A **self-similar group** G (e.g. Grigorchuk's group) is a group of automorphisms of the infinite, rooted binary tree $\mathcal{T}_d$ that is closed under restrictions.

Application to self-similar groups

A **self-similar group** G (e.g. Grigorchuk's group) is a group of automorphisms of the infinite, rooted binary tree $\mathcal{T}_d$ that is closed under restrictions.

Such a group acts by homeomorphisms on the d-ary Cantor set.

Application to self-similar groups

A **self-similar group** G (e.g. Grigorchuk's group) is a group of automorphisms of the infinite, rooted binary tree $\mathcal{T}_d$ that is closed under restrictions.

Such a group acts by homeomorphisms on the d-ary Cantor set.

Application to self-similar groups

A **self-similar group** G (e.g. Grigorchuk's group) is a group of automorphisms of the infinite, rooted binary tree $\mathcal{T}_d$ that is closed under restrictions.

Such a group acts by homeomorphisms on the d-ary Cantor set.

A **Röver–Nekrashevych group** $V_d G$ is a group generated by:

1. A self-similar group $G \leq \text{Aut}(\mathcal{T}_d)$, and
2. The Higman–Thompson group V_d.

Application to self-similar groups

A **self-similar group** G (e.g. Grigorchuk's group) is a group of automorphisms of the infinite, rooted binary tree $\mathcal{T}_d$ that is closed under restrictions.

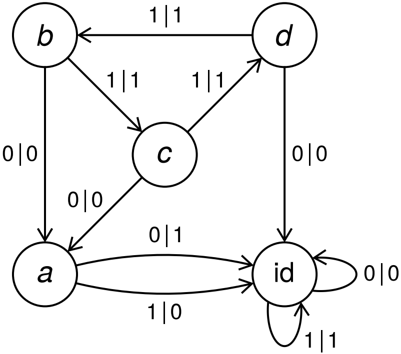
Such a group acts by homeomorphisms on the d-ary Cantor set.

A **Röver–Nekrashevych group** $V_d G$ is a group generated by:

1. A self-similar group $G \leq \mathrm{Aut}(\mathcal{T}_d)$, and
2. The Higman–Thompson group V_d.

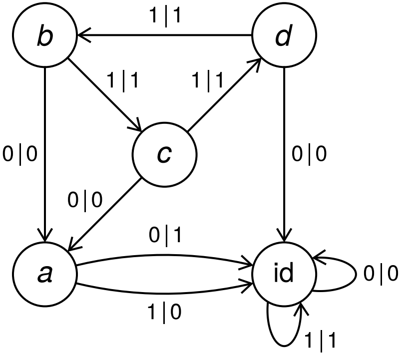
Applications to embeddings (Scott 1984, Röver 1999), C^*-algebras (Nekrashevych 2004), and finiteness properties (Skipper–Witzel–Zaremsky 2019).

Application to self-similar groups



The elements of Grigorchuk's group can be described by finite-state automata.

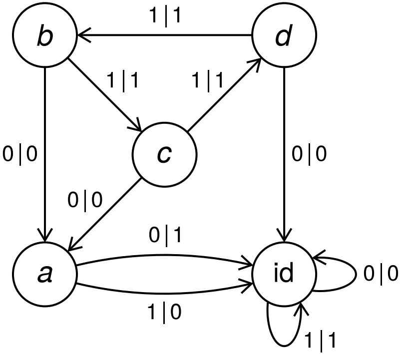
Application to self-similar groups



The elements of Grigorchuk's group can be described by finite-state automata.

A self-similar group G with this property is **rational**.

Application to self-similar groups

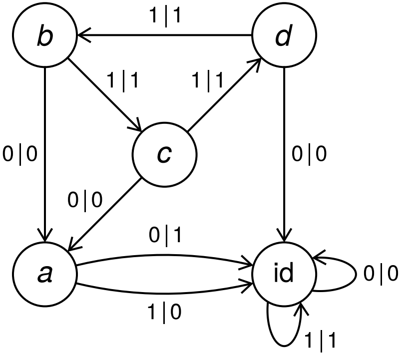


The elements of Grigorchuk's group can be described by finite-state automata.

A self-similar group *G* with this property is **rational**.

G is **contracting** if it has a finite nucleus of states.

Application to self-similar groups



The elements of Grigorchuk's group can be described by finite-state automata.

A self-similar group G with this property is **rational**.

G is **contracting** if it has a finite nucleus of states.

Theorem (Nekrashevych 2018)

If $G \leq \mathrm{Aut}(\mathcal{T}_d)$ is a contracting self-similar group, then $V_d G$ is finitely presented.

Application to self-similar groups

Theorem (Nekrashevych 2018)

If $G \leq \mathrm{Aut}(\mathcal{T}_d)$ *is a contracting self-similar group, then* $V_d G$ *is finitely presented.*

Application to self-similar groups

Theorem (Zaremsky 2022)

Let G be a group acting faithfully on a countable set X. Suppose:

1. *G is finitely presented,*
2. *The stabilizer of any finite subset of X is finitely generated, and*
3. *G is oligomorphic.*

Then G embeds into a finitely presented simple group.

Theorem (Nekrashevych 2018)

If $G \leq \mathrm{Aut}(\mathcal{T}_d)$ is a contracting self-similar group, then $V_d G$ is finitely presented.

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Sketch of Proof.

1. Embed every hyperbolic group into a “Thompson-like” group.
2. Embed these “Thompson-like” groups into finitely presented simple groups. □

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Sketch of Proof.

1. Embed every hyperbolic group into a "Thompson-like" group.
2. Embed these "Thompson-like" groups into finitely presented simple groups. ✓ □

Main Theorem

Main Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every hyperbolic group embeds into a finitely presented simple group.

Sketch of Proof.

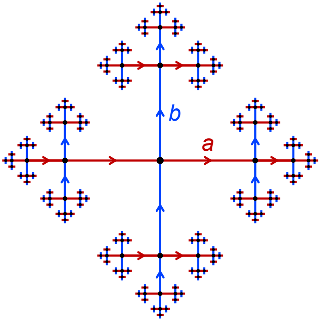
1. Embed every hyperbolic group into a "Thompson-like" group.
2. Embed these "Thompson-like" groups into finitely presented simple groups. ✓ □

For step #1, the "Thompson-like" group must satisfy the hypotheses of Zaremsky's theorem.

A Motivating Example

Example: The Free Group

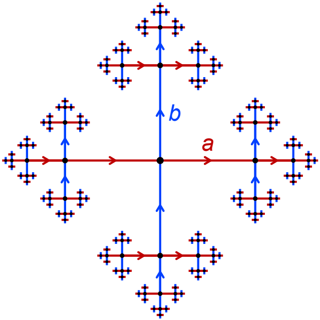
Let's embed the free group $F_2 = \langle a, b \rangle$ into a Thompson-like group.



Example: The Free Group

Let's embed the free group $F_2 = \langle a, b \rangle$ into a Thompson-like group.

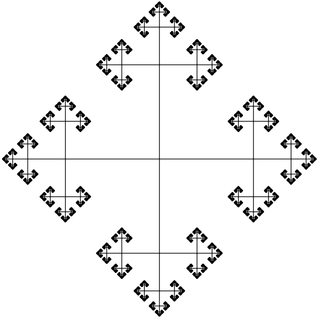
The boundary ∂F_2 is a Cantor space.



Example: The Free Group

Let's embed the free group $F_2 = \langle a, b \rangle$ into a Thompson-like group.

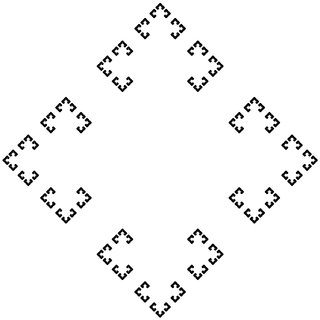
The boundary ∂F_2 is a Cantor space.



Example: The Free Group

Let's embed the free group $F_2 = \langle a, b \rangle$ into a Thompson-like group.

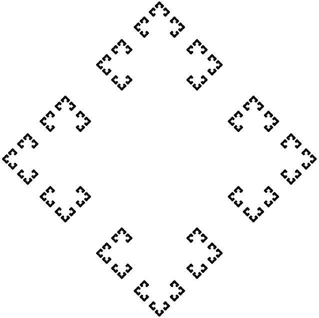
The boundary ∂F_2 is a Cantor space.



Example: The Free Group

Let's embed the free group $F_2 = \langle a, b \rangle$ into a Thompson-like group.

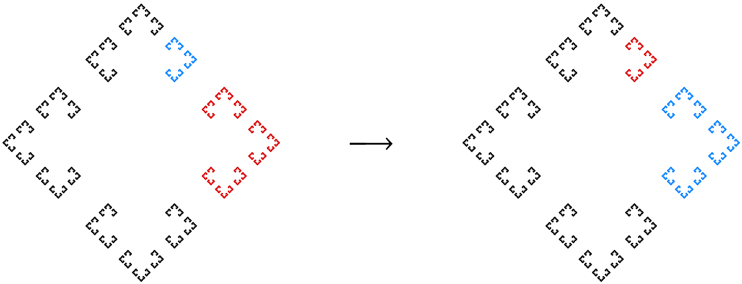
The boundary ∂F_2 is a Cantor space.



F_2 acts on ∂F_2 by homeomorphisms.

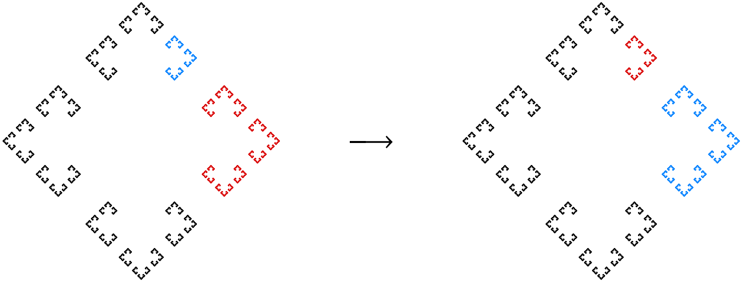
Example: The Free Group

Here is a homeomorphism of ∂F_2 which is *piecewise* in F_2:



Example: The Free Group

Here is a homeomorphism of ∂F_2 which is *piecewise* in F_2:



Let $[\![F_2 \mid \partial F_2]\!]$ be the group of all such homeomorphisms.

This is an example of a **full group**.

Full Groups

If X is a Cantor space and $G \leq \mathrm{Homeo}(X)$, let

$$[\![G \mid X]\!]$$

be the group of all piecewise-G homeomorphisms of X.

Full Groups

If X is a Cantor space and $G \leq \text{Homeo}(X)$, let

$$[\![G \mid X]\!]$$

be the group of all piecewise-G homeomorphisms of X.

$[\![G \mid X]\!]$ is the **full closure** of G, and G is **full** if $[\![G \mid X]\!] = G$.

Full Groups

If X is a Cantor space and $G \leq \mathrm{Homeo}(X)$, let

$$[\![G \mid X]\!]$$

be the group of all piecewise-G homeomorphisms of X.

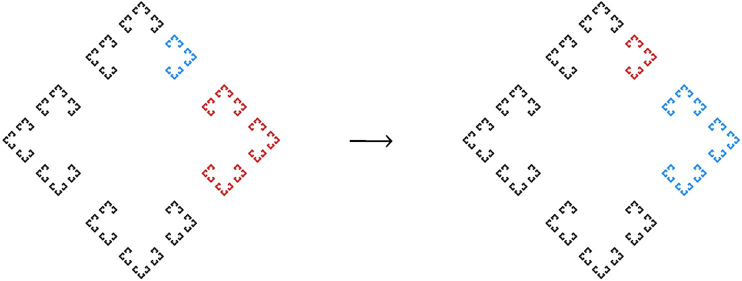
$[\![G \mid X]\!]$ is the **full closure** of G, and G is **full** if $[\![G \mid X]\!] = G$.

Examples of Full Groups

1. Higman–Thompson groups $V_{d,r}$.
2. Stein groups $V_{\{d_1,\ldots,d_n\},r}$.
3. Brin–Thompson groups nV.
4. Röver–Nekrashevych groups $V_d G$.

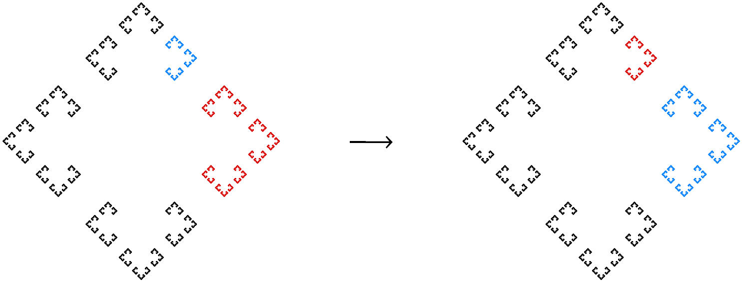
Example: The Free Group

The full group $[\![F_2 \mid \partial F_2]\!]$ contains F_2, and is very Thompson-like.



Example: The Free Group

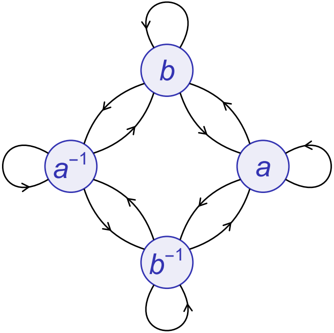
The full group $[\![F_2 \mid \partial F_2]\!]$ contains F_2, and is very Thompson-like.



(Matui 2015) It is finitely presented (type F_∞), has simple commutator subgroup, and its abelianization is $\mathbb{Z}^2$.

Example: The Free Group

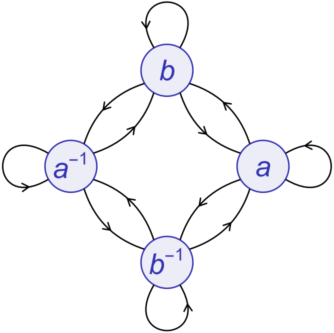
The boundary ∂F_2 is a **subshift of finite type**.



It consists of all infinite directed paths in the graph Γ above.

Example: The Free Group

The boundary ∂F_2 is a **subshift of finite type**.



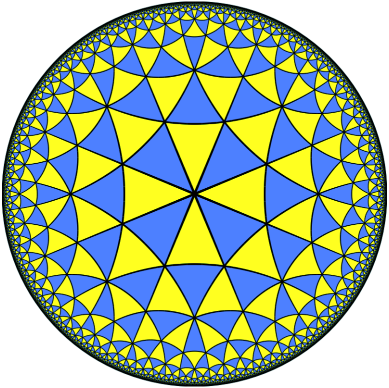
It consists of all infinite directed paths in the graph Γ above.

(Matsumoto 2015, Matui 2015) Each irreducible subshift of finite type has an associated (Thompson-like) full group V_Γ.

Hyperbolic Groups

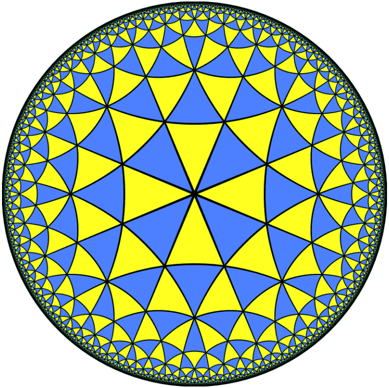
Hyperbolic Boundaries

The Gromov boundary ∂G of a hyperbolic group G isn't usually a Cantor space.



Hyperbolic Boundaries

The Gromov boundary ∂G of a hyperbolic group G isn't usually a Cantor space.



But the **horofunction boundary** $\partial_h G$ of G usually *is* a Cantor space.

The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

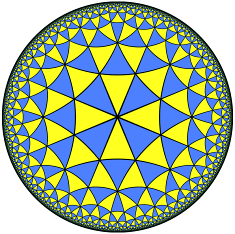
The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

Theorem (Webster–Winchester 2005)

If G is hyperbolic, then ∂G is a quotient of $\partial_h G$, and the quotient map $\partial_h G \longrightarrow \partial G$ is finite-to-one.



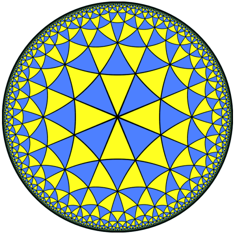
The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

Theorem (Webster–Winchester 2005)

If G is hyperbolic, then ∂G is a quotient of $\partial_h G$, and the quotient map $\partial_h G \to \partial G$ is finite-to-one. (See Perego 2023.)



The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

Theorem (Webster–Winchester 2005)

If G is hyperbolic, then ∂G is a quotient of $\partial_h G$, and the quotient map $\partial_h G \longrightarrow \partial G$ is finite-to-one. (See Perego 2023.)

The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

Theorem (Webster–Winchester 2005)

If G is hyperbolic, then ∂G is a quotient of $\partial_h G$, and the quotient map $\partial_h G \to \partial G$ is finite-to-one. (See Perego 2023.)

Some Technical Problems

1. G might not act faithfully on $\partial_h G$.
2. $\partial_h G$ might not be a Cantor space.

The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

Theorem (Webster–Winchester 2005)

If G is hyperbolic, then ∂G is a quotient of $\partial_h G$, and the quotient map $\partial_h G \to \partial G$ is finite-to-one. (See Perego 2023.)

Some Technical Problems

1. G might not act faithfully on $\partial_h G$.
2. $\partial_h G$ might not be a Cantor space.

Both can be solved by first embedding G in $G * \mathbb{Z}$.

The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

Theorem (Webster–Winchester 2005)

If G is hyperbolic, then ∂G is a quotient of $\partial_h G$, and the quotient map $\partial_h G \longrightarrow \partial G$ is finite-to-one. (See Perego 2023.)

The horofunction boundary

Every f.g. group G has a **horofunction boundary** $\partial_h G$, which is compact, totally disconnected, and metrizable.

G acts on $\partial_h G$ by homeomorphisms.

Theorem (Webster–Winchester 2005)

If G is hyperbolic, then ∂G is a quotient of $\partial_h G$, and the quotient map $\partial_h G \to \partial G$ is finite-to-one. (See Perego 2023.)

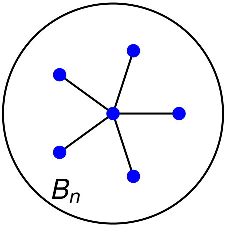
Assuming $\partial_h G$ is well-behaved (i.e. a Cantor space on which G acts faithfully), we get an embedding of G in the "Thompson-like" group $[\![G \mid \partial_h G]\!]$.

Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.

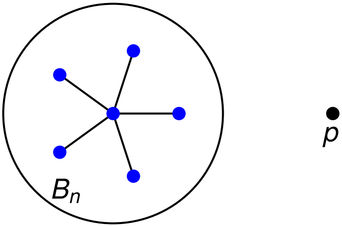
Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.



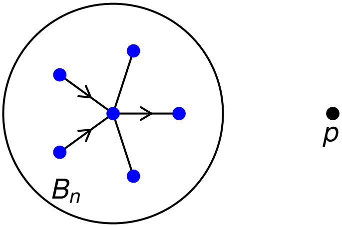
Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.



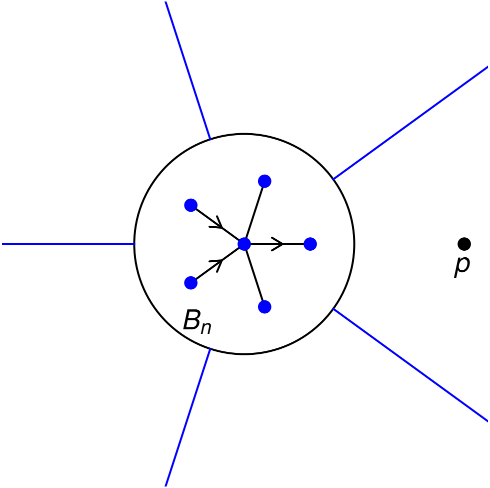
Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.



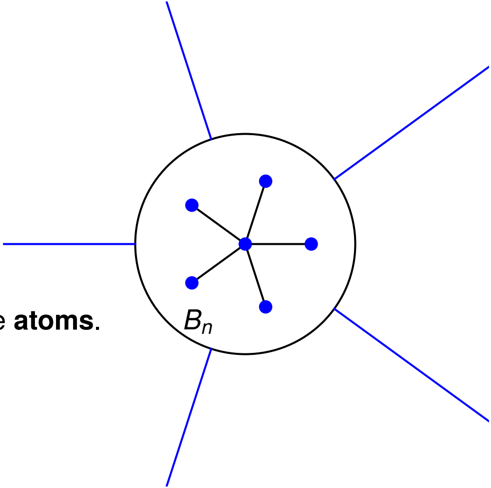
Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.



Defining $\partial_h G$ using atoms

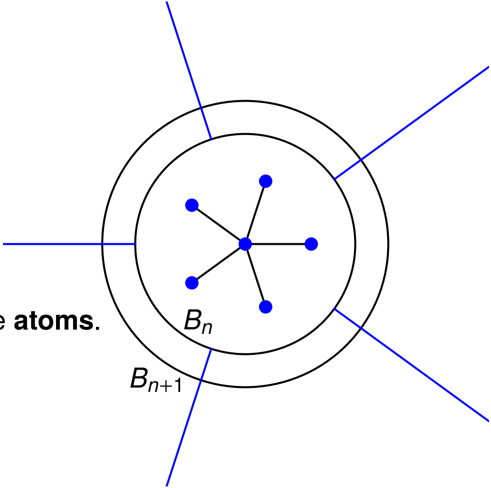
The horofunction boundary $\partial_h G$ can be defined as follows.



These are the **atoms**.

Defining $\partial_h G$ using atoms

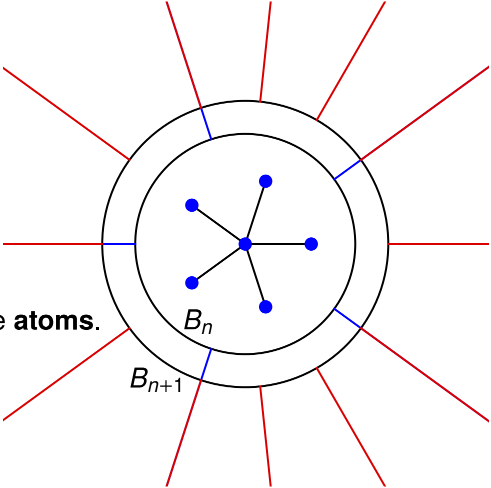
The horofunction boundary $\partial_h G$ can be defined as follows.



These are the **atoms**.

Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.



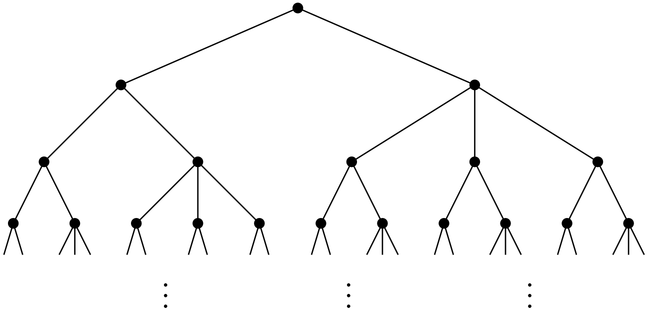
These are the **atoms**.

Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.

Defining $\partial_h G$ using atoms

The horofunction boundary $\partial_h G$ can be defined as follows.



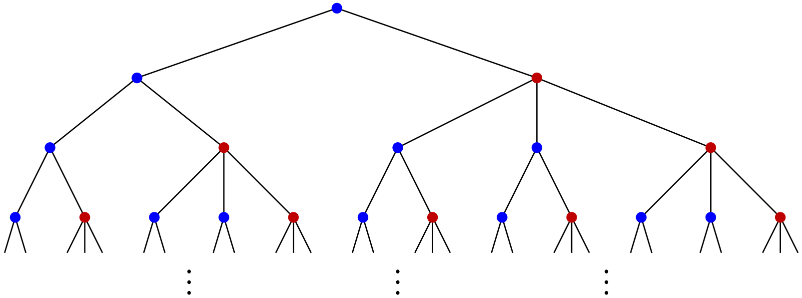
(B–Bleak–Matucci 2021)

This defines the **tree of atoms**. Its space of ends is $\partial_h G$.

Atoms in hyperbolic groups

Theorem (B–Bleak–Matucci 2021)

For a hyperbolic group G, the tree of atoms is a self-similar tree.

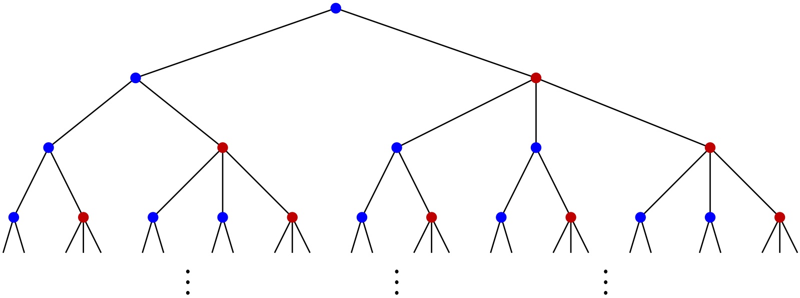


Atoms in hyperbolic groups

Theorem (B–Bleak–Matucci 2021)

For a hyperbolic group G, the tree of atoms is a self-similar tree.

This identifies $\partial_h G$ with a clopen set in some subshift of finite type.



Atoms in hyperbolic groups

Theorem (B–Bleak–Matucci 2021)

For a hyperbolic group G, the tree of atoms is a self-similar tree.

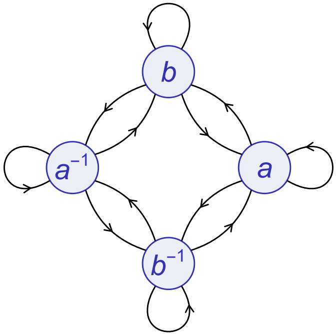
This identifies $\partial_h G$ with a clopen set in some subshift of finite type.

Atoms in hyperbolic groups

Theorem (B–Bleak–Matucci 2021)

For a hyperbolic group G, the tree of atoms is a self-similar tree.

This identifies $\partial_h G$ with a clopen set in some subshift of finite type.

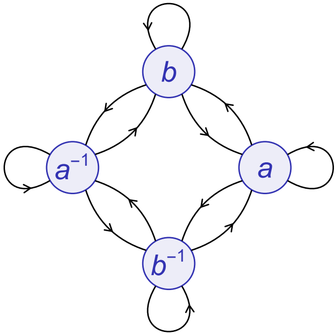


Atoms in hyperbolic groups

Theorem (B–Bleak–Matucci 2021)

For a hyperbolic group G, the tree of atoms is a self-similar tree.

This identifies $\partial_h G$ with a clopen set in some subshift of finite type.



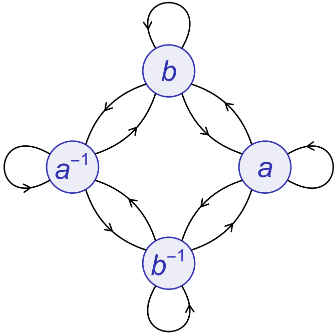
But what does the action of G on this clopen set look like?

Atoms in hyperbolic groups

Theorem (B–Bleak–Matucci 2021)

For a hyperbolic group G, the tree of atoms is a self-similar tree.

This identifies $\partial_h G$ with a clopen set in some subshift of finite type.



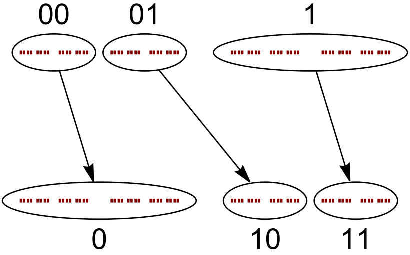
But what does the action of G on this clopen set look like?

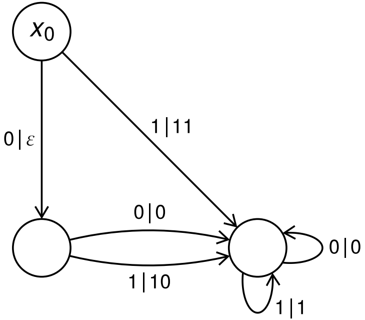
Answer: Finite-state automata.

Asynchronous automata

(Grigorchuk, Nekrashevych, Sushchanskiĭ 2000)

Elements of V_d can be described by **asynchronous automata**.

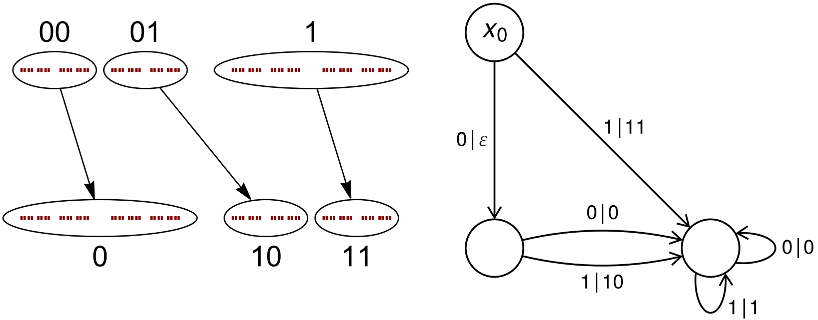




Asynchronous automata

(Grigorchuk, Nekrashevych, Sushchanskiĭ 2000)

Elements of V_d can be described by **asynchronous automata**.



If G is a rational self-similar group, then elements of $V_d G$ can also be described by asynchronous automata.

Asynchronous automata

Theorem (B–Bleak–Matucci 2021)

If G is a hyperbolic group, then G acts by asynchronous automata (w.r.t. the subshift) on $\partial_h G$.

Asynchronous automata

Theorem (B–Bleak–Matucci 2021)

If G is a hyperbolic group, then G acts by asynchronous automata (w.r.t. the subshift) on $\partial_h G$.

Indeed, the full group $[\![G \mid \partial_h G]\!]$ acts by asynchronous automata and contains V_Γ. We call a group with these properties a **full rational similarity group (full RSG)**.

Asynchronous automata

Theorem (B–Bleak–Matucci 2021)

If G is a hyperbolic group, then G acts by asynchronous automata (w.r.t. the subshift) on $\partial_h G$.

Indeed, the full group $[\![G \mid \partial_h G]\!]$ acts by asynchronous automata and contains V_Γ. We call a group with these properties a **full rational similarity group (full RSG)**.

So the class of full RSG's includes:

1. Röver–Nekrashevych groups $V_d G$, where G is any rational self-similar group.
2. $[\![G \mid \partial_h G]\!]$ for any well-behaved hyperbolic group G.

Full, contracting RSG's

A full RSG is **contracting** if it has a finite nucleus of states.

Full, contracting RSG's

A full RSG is **contracting** if it has a finite nucleus of states.

Theorem (B–Bleak–Matucci–Zaremsky 2023)

If G is hyperbolic group with well-behaved $\partial_h G$, then $[\![G \mid \partial_h G]\!]$ is a full, contracting RSG.

Full, contracting RSG's

A full RSG is **contracting** if it has a finite nucleus of states.

Theorem (B–Bleak–Matucci–Zaremsky 2023)

If G is hyperbolic group with well-behaved $\partial_h G$, then $[\![G \mid \partial_h G]\!]$ is a full, contracting RSG.

Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every full, contracting RSG is finitely presented.

Full, contracting RSG's

A full RSG is **contracting** if it has a finite nucleus of states.

Theorem (B–Bleak–Matucci–Zaremsky 2023)

If G is hyperbolic group with well-behaved $\partial_h G$, then $[\![G \mid \partial_h G]\!]$ is a full, contracting RSG.

Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every full, contracting RSG is finitely presented.

Theorem (B–Bleak–Matucci–Zaremsky 2023)

Every full, contracting RSG embeds into a finitely presented simple group.

Questions

For which hyperbolic groups G is $\partial_h G$ well-behaved?

- Is $\partial_h G$ always well-behaved if G acts faithfully on ∂G?
- If G is non-elementary, is $\partial_h G$ always well-behaved with respect to some generating set?

What can be said about the finiteness properties of $[\![G \mid \partial_h G]\!]$?

Are there non-hyperbolic groups G for which $[\![G \mid \partial_h G]\!]$ finitely presented? Can we get any more Boone–Higman embeddings this way?

- If $G = \mathbb{Z}^2$, then $[\![G \mid \partial_h G]\!] \cong H_2 \times H_2 \times H_2 \times H_2$.
- If we want $[\![G \mid \partial_h G]\!]$ to be contracting, the groups of germs must be virtually cyclic.