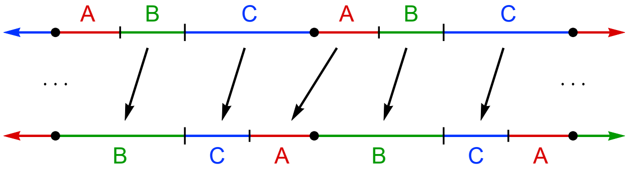
Embeddings into Finitely Presented Simple Groups



Jim Belk, University of Glasgow

AI@Bicocca, 13 December 2022

The Boone–Higman Conjecture

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Let G be a finitely generated group. Then:

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Here a group has ***solvable word problem*** if there exists an algorithm to determine whether a given word in the generators represents the identity.

Theorem (Novikov 1955, Boone 1958)

There exist finitely presented groups with unsolvable word problem.

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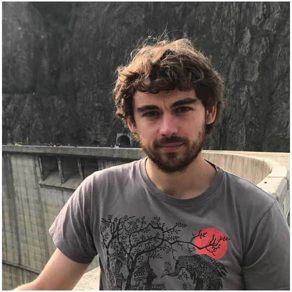
Recent progress: Many groups of interest embed into finitely presented simple groups.

Collaborators

Collin Bleak

University of St Andrews



James Hyde

University of Copenhagen

Collaborators

Francesco Matucci

University of Milano–Bicocca

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Higman's Embedding Theorem

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A countable group presentation

$$\langle s_1, s_2, s_3, \ldots \mid r_1, r_2, r_3, \ldots \rangle$$

is ***computable*** if there exists an algorithm that outputs the list of relations.

A group is ***computably presented*** if it admits such a presentation.

Examples

1. Any finitely presented group.
2. Any finitely generated subgroup of a finitely presented group.

Higman's Embedding Theorem (1961)

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Graham Higman, 1960

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1. *Countably generated groups with a computable presentations.*

   Follows from Higman–Neumann–Neumann 1949.

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1. *Countably generated groups with a computable presentations.*
2. *Countable abelian groups.*

   Since every such group embeds in $\bigoplus_\omega \mathbb{Q} \oplus \bigoplus_\omega \mathbb{Q}/\mathbb{Z}$.

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Problem (Higman): Find an explicit and natural example of a finitely presented group that contains $\mathbb{Q}$.

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This theorem has the form

G has a certain algorithmic property	$\Leftrightarrow$	G embeds into a certain kind of group

Question (Higman): Are there other theorems of this type?

The Boone–Higman Conjecture

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Richard J. Thompson, 2004

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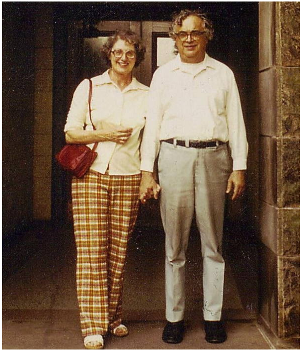
Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.



William and Eileen Boone, 1979

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Proof.

Given a presentation $\langle s_1, \dots s_m \mid r_1, \dots r_n \rangle$ for a simple group G and a word w, we run two simultaneous searches:

Search #1	**Search #2**
Search for a proof that $$w = 1$$ using the relations $r_1, \dots, r_n$.	Search for a proof that $$s_1 = \cdots = s_m = 1$$ using $w = 1$ and $r_1, \dots, r_n$.

Eventually one of the searches terminates. □

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Boone and Higman recognized Thompson's observation as a group-theoretic analogue of a basic observation in logic:

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complete theory	simple group
decidable theory	decidable word problem

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Note: By a result of Clapham (1965), it would suffice to prove the conjecture for finitely presented groups.

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Like Higman's embedding theorem, this statement has the form

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As a corollary, the following groups would also embed into finitely presented simple groups:

1. Any computably presented group with solvable word problem.
2. Any countable abelian group.

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Trick: Given words $u, v \neq_G 1$, consider the group

$$G' = \left\langle G, x, t \,\middle|\, (uu^x)^t = u^x v \right\rangle.$$

G' is an HNN extension of $G * \langle x \rangle$, so G embeds into G'.

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But now v lies in the normal closure of u.

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Sketch of Proof. Let

$$\sigma(G) = \left\langle G, x, t_1, t_2, \ldots \,\middle|\, (u_i u_i^x)^{t_i} = u_i^x v_i \right\rangle$$

where (u_i, v_i) is an enumeration of *all* pairs of non-identity words in G.

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The desired simple group is the union of the sequence

$$G \;\leq\; \sigma(G) \;\leq\; \sigma^2(G) \;\leq\; \sigma^3(G) \;\leq\; \cdots. \qquad \square$$

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Theorem (Thompson 1980)

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Theorem (Sacerdote 1977)

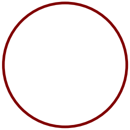
There are analogues of Boone and Higman's theorem for the order, conjugacy, power, and subgroup membership problems.

Finitely Presented Simple Groups

Thompson's Groups

In 1965, Richard J. Thompson defined three infinite groups.

F acts on the interval.



T acts on the circle.

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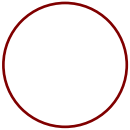

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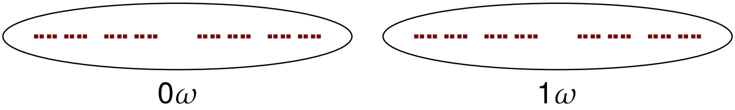

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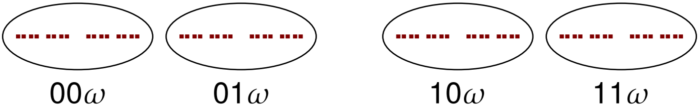

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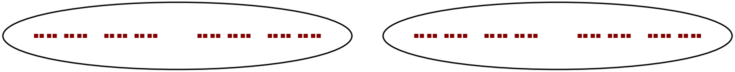

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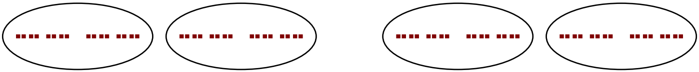

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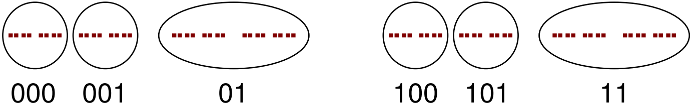

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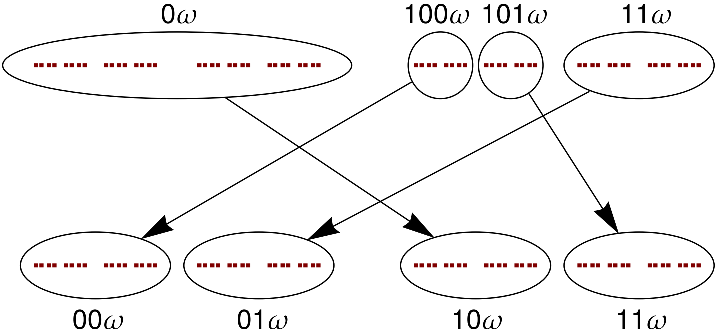
Definition of V

Thompson's group V is the group of all homeomorphisms that map "linearly" between the pieces of two dyadic subdivisions.



This group V is finitely presented and simple.

Thompson's Groups

V acts by homeomorphisms on the Cantor set.

F and *T* are subgroups of *V*.

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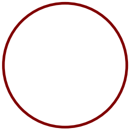

T is the subgroup of V that preserves the circular order.

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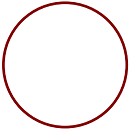

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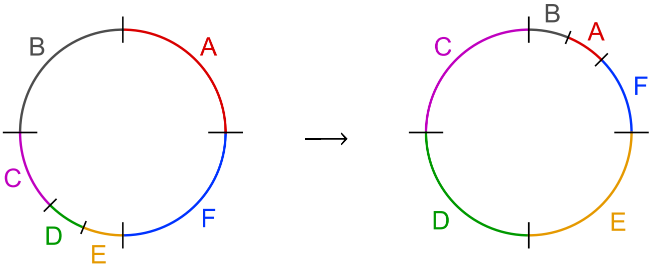
Thompson's Group T

For example, here is an element of Thompson's group T.



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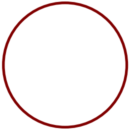

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Subgroups of V

The following groups embed into V:

1. All finite groups, free groups, and free abelian groups.
2. (Higman 1974) Countable locally finite groups.
3. (Higman 1974, Brown 1987) Generalized Thompson groups F_n, T_n, and V_n.
4. (Röver 1999) Free products of finitely many finite groups.
5. (Guba–Sapir 1999, Bleak 2008) Many solvable groups.
6. (Bleak–Kassabov–Matucci 2011) $\mathbb{Q}/\mathbb{Z}$.

Grigorchuk's Group

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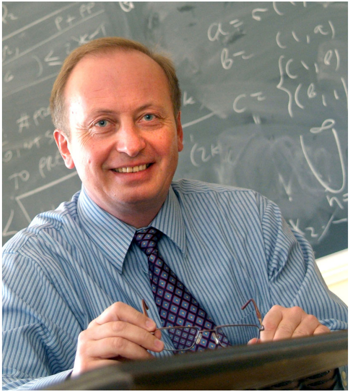
Grigorchuk (1979) defined a certain finitely generated group $\mathcal{G} \leq \mathrm{Aut}(T_2)$.



Rostislav Grigorchuk

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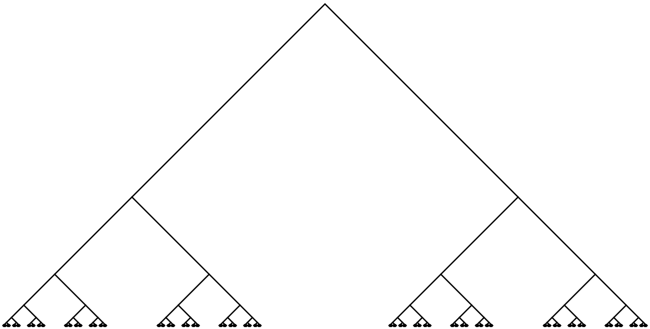

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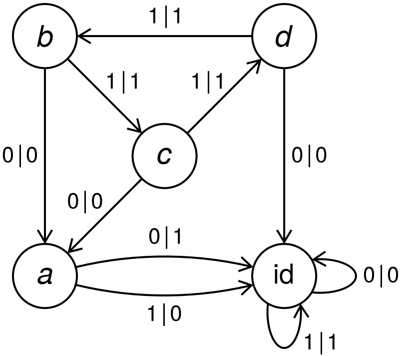
Properties of $\mathcal{G}$ (Grigorchuk 1979 and 1984)

- $\mathcal{G}$ is a solution to the Burnside problem: it is infinite and finitely generated, and every element has finite order.
- $\mathcal{G}$ has intermediate growth: the number of elements of length less than n grows like $\exp\left(n^{0.7675}\right)$ (Erschler–Zheng 2020).

Grigorchuk's Group

The action of $\mathcal{G}$ on binary sequences in $\{0,1\}^\omega$ can be described by ***automata***.



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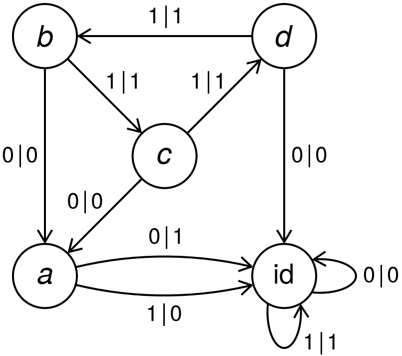

$c(1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \cdots)$

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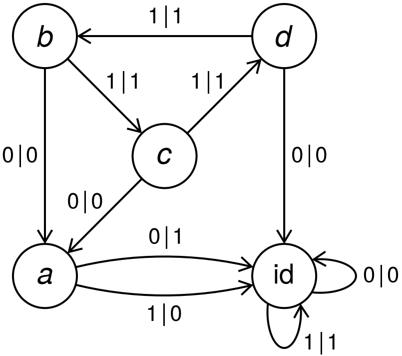

$$c(1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \cdots)$$
$$= 1 \cdot d(1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \cdots)$$

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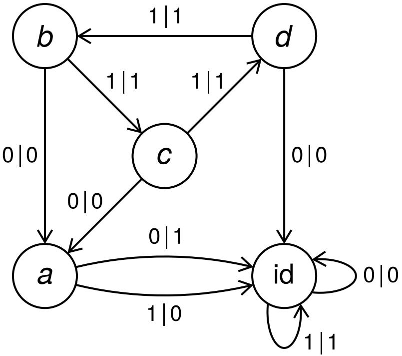

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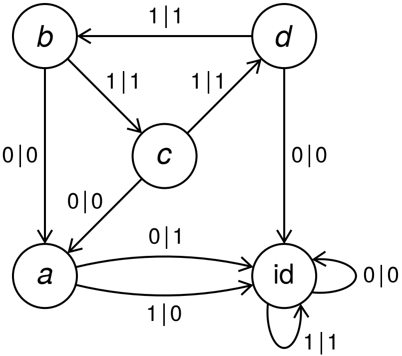

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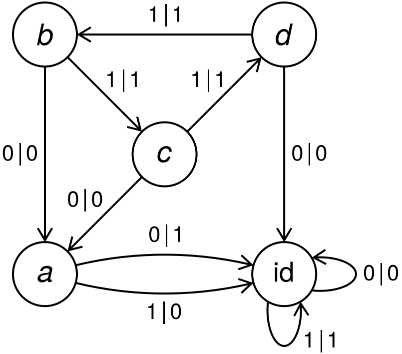

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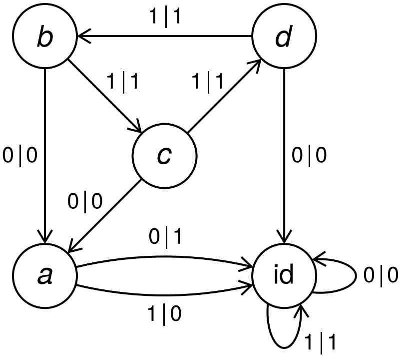

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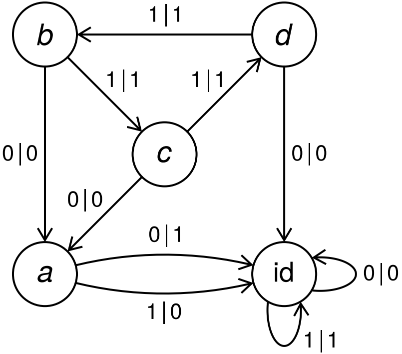

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Every element of $\mathcal{G}$ has such an automaton.

Grigorchuk's Group

Does $\mathcal{G}$ embed into Thompson's group V?

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Theorem (Röver 1999)

Yes. The group $V\mathcal{G}$ generated by V and $\mathcal{G}$ is finitely presented and simple!

Nekrashevych Groups

Nekrashevych (2005) generalized Grigorchuk's group to the family of ***contracting self-similar groups*** $G \leq \mathrm{Aut}(T_d)$.

Volodymyr Nekrashevych

Nekrashevych Groups

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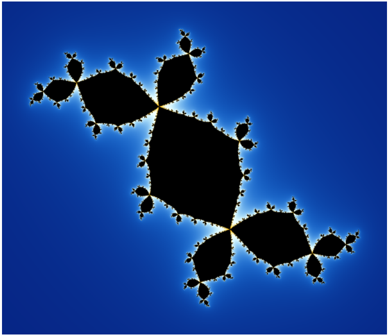
These include the "iterated monodromy groups" associated to complex dynamical systems.



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Theorem (Nekrashevych 2013)

Every Nekrashevych group $V_d G$ is finitely presented.

Nekrashevych also gave necessary and sufficient conditions for $V_d G$ to be simple.

This gives Boone–Higman embeddings for *some* contracting self-similar groups.

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Matthew Brin

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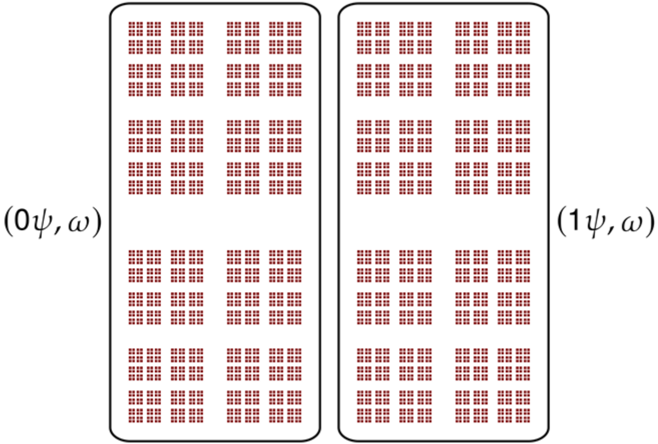

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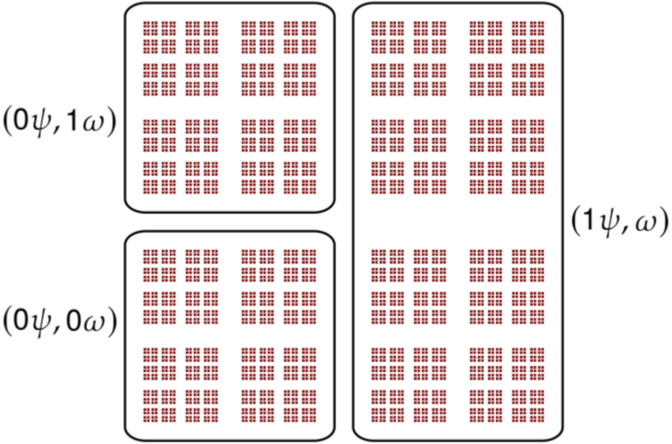

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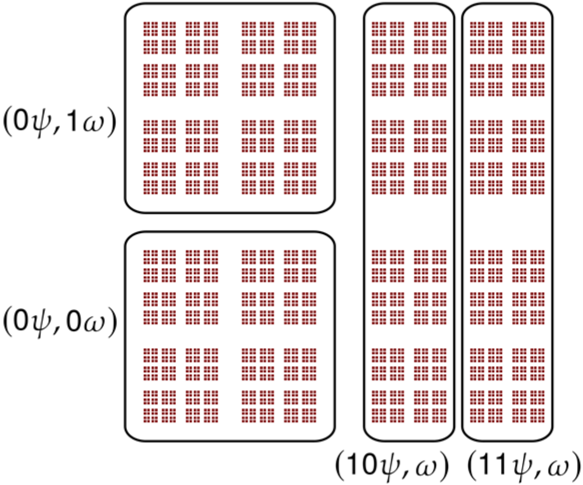
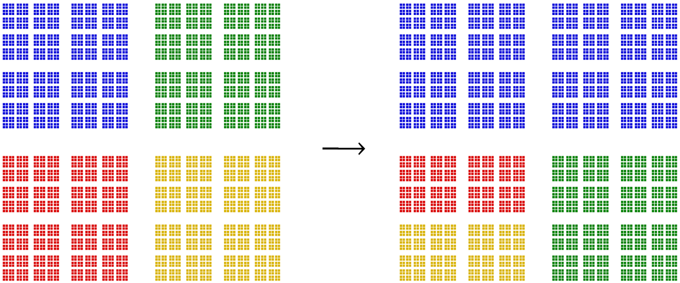


Brin's Groups

Elements of $2V$ map "linearly" between two subdivisions.



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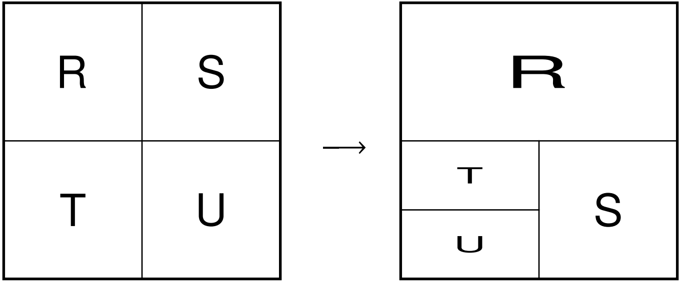

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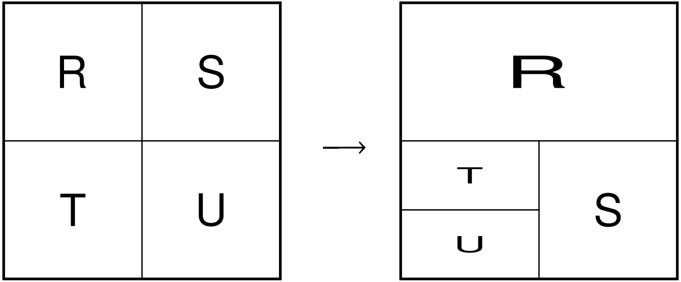


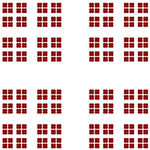
Theorem (Brin 2004)

The group $2V$ is finitely presented and simple.

Brin's Groups

Brin defined a family of groups nV ($n \geq 1$) similarly, with $1V = V$.

V



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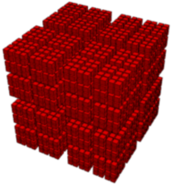

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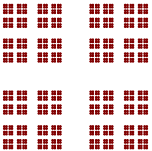

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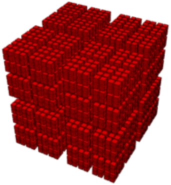

$3V$

$\cdots$

Theorem (Brin 2009)

The group nV is finitely presented and simple for all $n \geq 1$.

Brin's Groups

These groups have very interesting algorithmic properties.

Theorem (B–Bleak 2014)

The order problem in nV *is unsolvable for* $n \geq 2$

Theorem (B–Bleak–Matucci 2016)

The subgroup membership problem in nV *is unsolvable for* $n \geq 2$.

Theorem (Salo 2020)

The conjugacy problem in nV *is unsolvable for* $n \geq 2$.

Right-angled Artin groups

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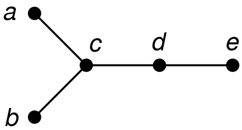
Given a finite graph Γ



the corresponding **right-angled Artin group (RAAG)** has one generator for each vertex, with edges corresponding to generators that commute:

$$\left\langle a, b, c, d, e \;\middle|\; ac = ca,\ bc = cb,\ cd = dc,\ de = ed \right\rangle.$$

Embeddings into RAAG's

Many groups either embed or virtually embed into a RAAG:

1. (Wise 2009) All limit groups.
2. (Haglund–Wise 2010) All finitely generated Coxeter groups.
3. (Agol 2012) All cubulated hyperbolic groups.
4. (Agol 2012) Fundamental groups of finite-volume hyperbolic 3-manifolds.
5. (Przytycki–Wise 2012) Fundamental groups of compact Riemannian 3-manifolds of non-positive curvature.

Boone–Higman for RAAG's

Theorem (B–Bleak–Matucci 2016)

If a group G virtually embeds into a RAAG, then G embeds into one of Brin's groups nV.

Boone–Higman for RAAG's

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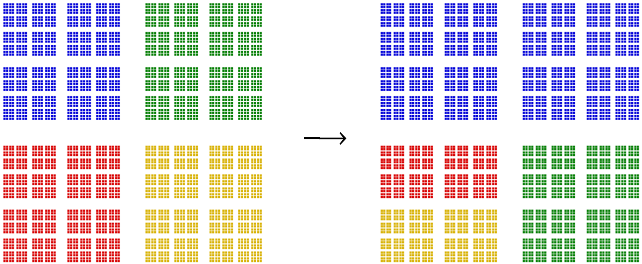
Salo (2021) has shown that all RAAG's embed into $2V$.



Countable Abelian Groups

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In 1999, Martin Bridson and Pierre de la Harpe submitted this question to the Kourovka notebook as a “well-known” problem.

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In 2020, James Hyde, Francesco Matucci, and I noticed an elementary solution.

Countable Abelian Groups

Recall that Thompson's group T acts on S^1.

A ***lift*** of an element $g \in T$ is a homeomorphism $\overline{g} : \mathbb{R} \to \mathbb{R}$ that makes the following diagram commute:

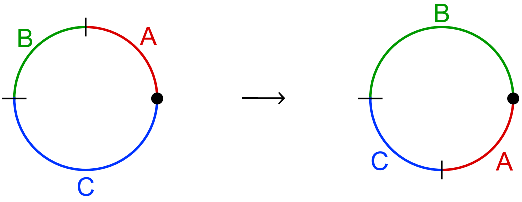
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\overline{g}} & \mathbb{R} \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow[g]{} & S^1 \end{array}$$

Note: If $\overline{g}$ is a lift of g then so is $\overline{g} + n$ for any $n \in \mathbb{Z}$.

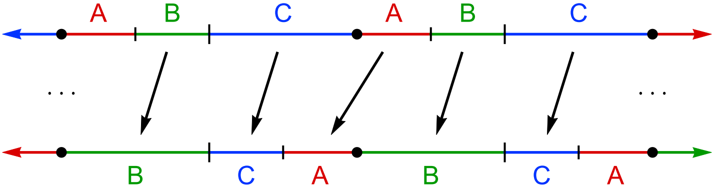
Let $\overline{T}$ be the group of all lifts of elements of T.

Countable Abelian Groups

For example, here's an element of T:



and here's one possible lift in $\overline{T}$:



Countable Abelian Groups

Theorem (B–Hyde–Matucci 2020)

The group $\overline{T}$ is finitely presented and contains $\mathbb{Q}$.

$$\overline{T} = \langle a, b \mid a^4 b^{-3}, (ba)^5 b^{-9}, [bab, a^2 baba^2], [bab, a^2 b^2 a^2 baba^2 ba^2] \rangle$$

Note: We did not introduce this group $\overline{T}$. It had previously appeared in the work of Ghys and Sergiescu (1987).

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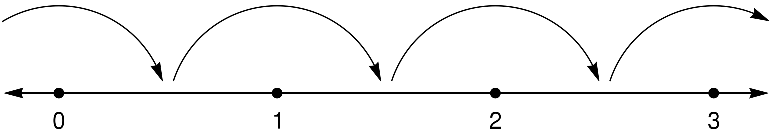

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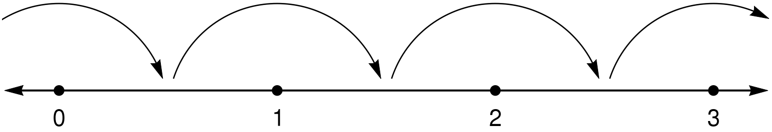
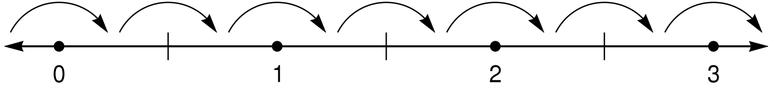


It's easy to find a square root f_2 of f_1:



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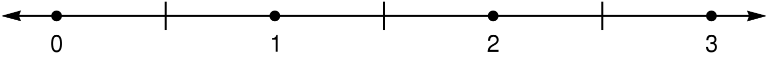
Proof. Now construct a cube root f_3 of f_2:



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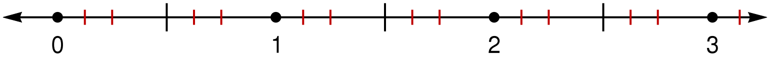

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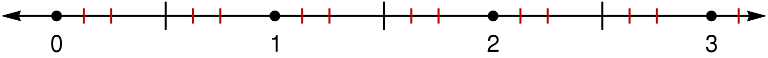

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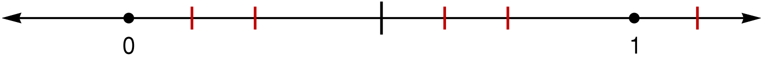

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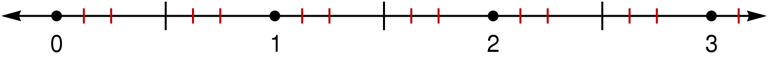

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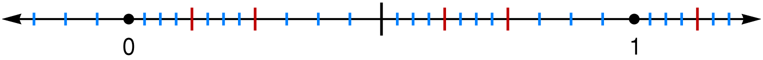

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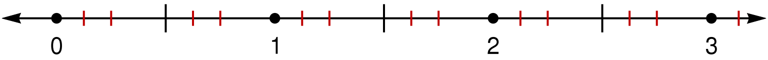

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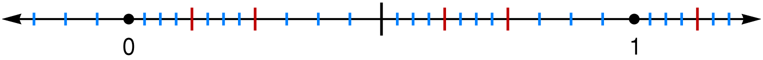

Then $\langle f_1, f_2, f_3, f_4, \ldots \rangle \cong \mathbb{Q}$. □

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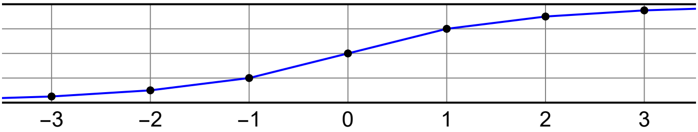

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We prove that the group $V\overline{T}$ generated by V and $\overline{T}$ is finitely presented, simple, and contains $\bigoplus_{\omega}\mathbb{Q} \oplus \bigoplus_{\omega}\mathbb{Q}/\mathbb{Z}$. □

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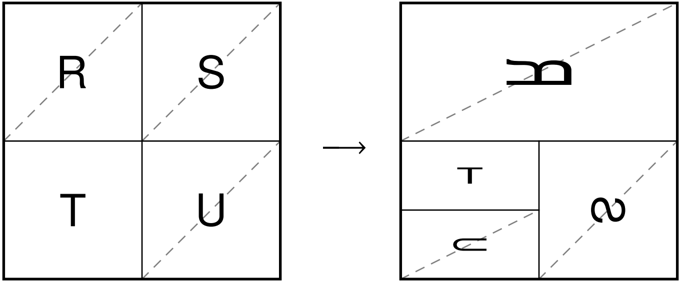

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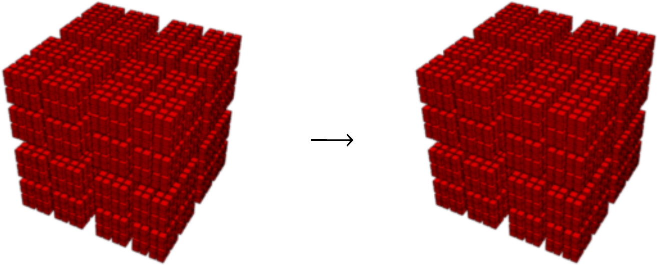

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Theorem (B–Zaremsky 2020)

Any twisted ωV is simple, and is finitely generated as long as the action of G on X is transitive.

Twisting Brin's Groups

In 2020, Matthew Zaremsky and I considered a "twisted" version of Brin's group $2V$.

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Corollary (B–Zaremsky 2020)

Any finitely generated group G embeds isometrically into a finitely generated simple group.

Twisting Brin's Groups

We can also get finitely presented simple groups.

Theorem (B–Zaremsky 2020, Zaremsky 2022)

Suppose:

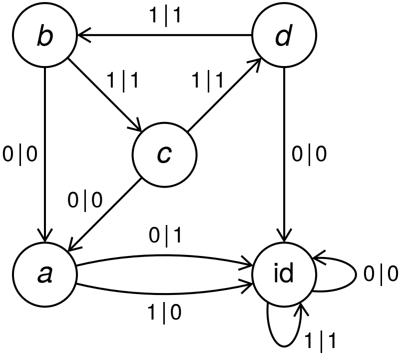
1. *G is finitely presented,*
2. *G acts highly transitively on a set X, and*
3. *Stabilizers of finite subsets of X are finitely generated.*

Then the resulting twisted ωV is a finitely presented simple group that contains G.

Twisting Brin's Groups

Corollary

Every contracting self-similar embeds into a finitely presented simple group.



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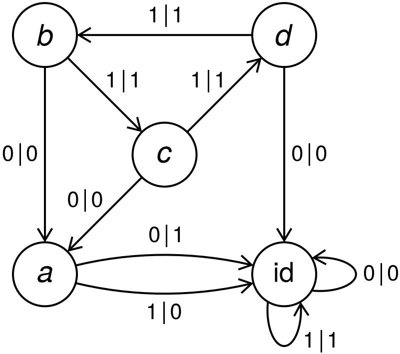

Sketch of Proof.

The Nekrashevych group $V_d G$ is finitely presented, highly transitive on any orbit, and has finitely generated stabilizers, so the resulting twisted ωV is finitely presented and simple.

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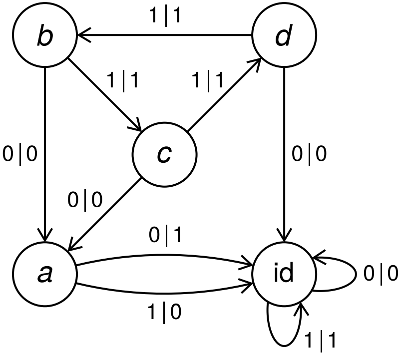

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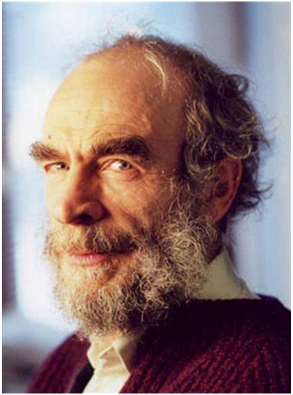
We can similarly handle many other "Thompson-like" groups.

Hyperbolic Groups

Hyperbolic Groups

Gromov (1987) defined a finitely generated group to be ***hyperbolic*** if its Cayley graph satisfies the δ-thin triangles condition.



Misha Gromov

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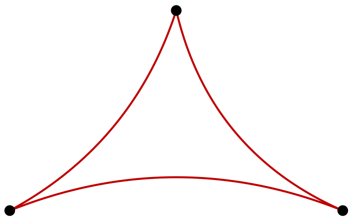

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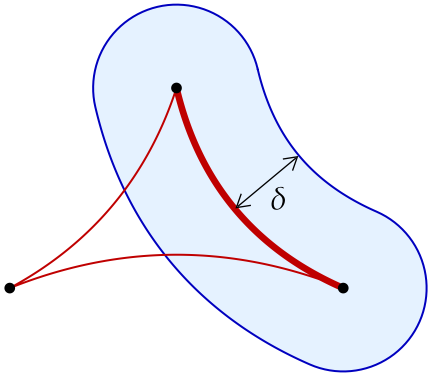

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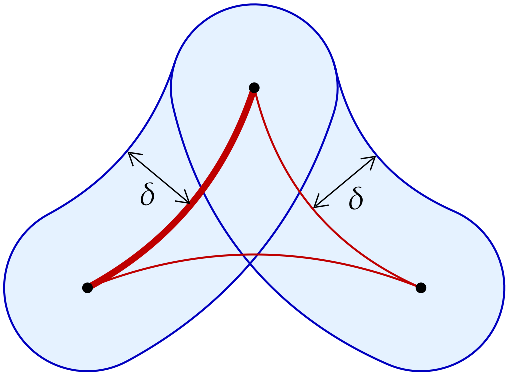

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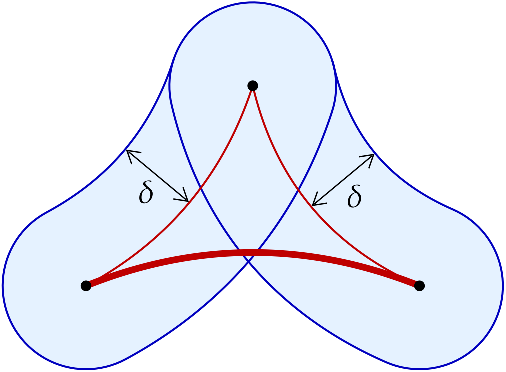

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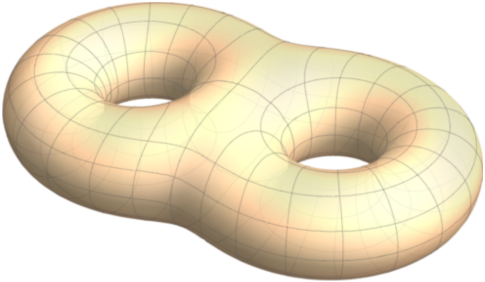
For example, the fundamental group of any compact hyperbolic manifold is a hyperbolic group.



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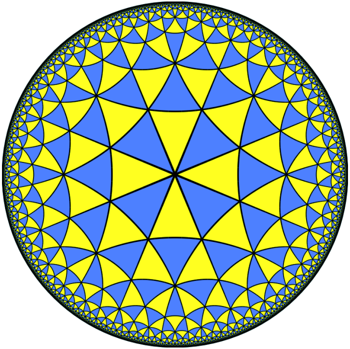
In a certain precise sense, almost every finitely presented group is hyperbolic (Ol'Shanskii 1991).

Hyperbolic Groups

Theorem (B–Bleak–Matucci–Zaremsky 2022)

Every hyperbolic group G embeds into a finitely presented simple group.



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4. Conclude that $V[G]$ embeds into a twisted ωV which is finitely presented and simple.

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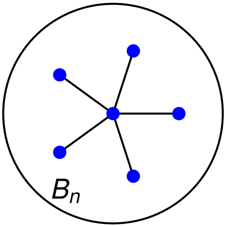

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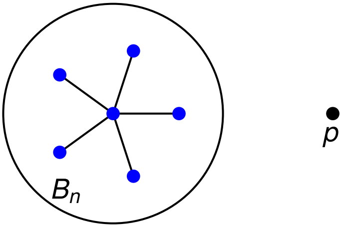

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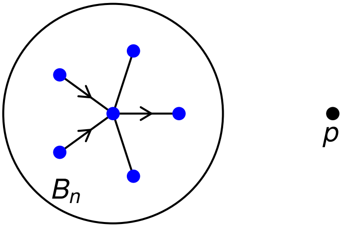

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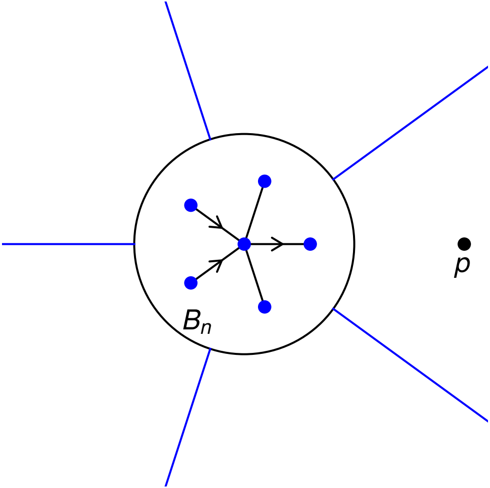

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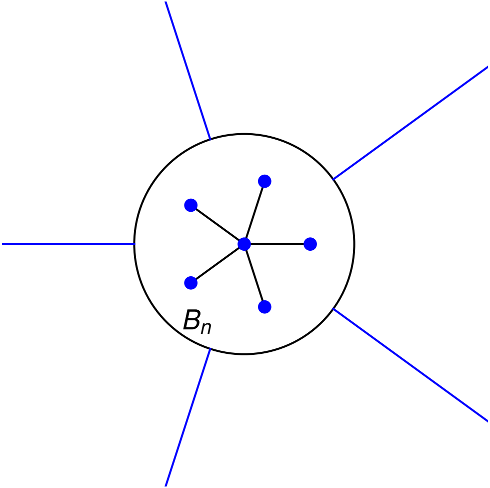

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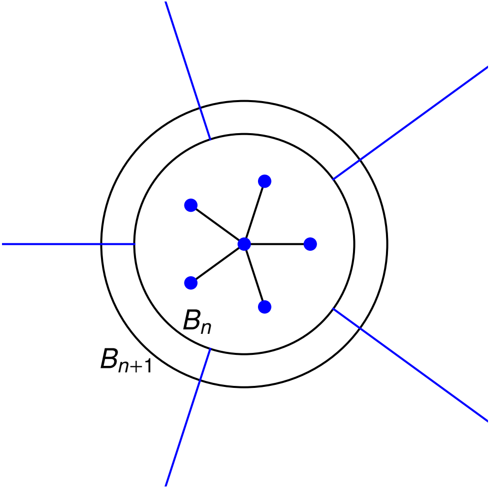

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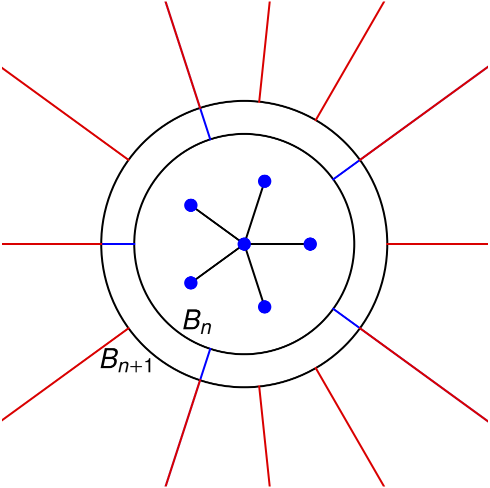

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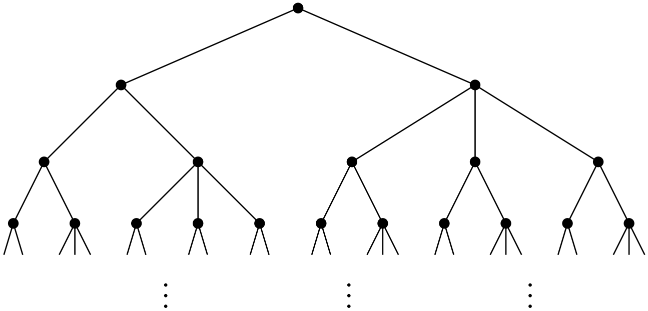


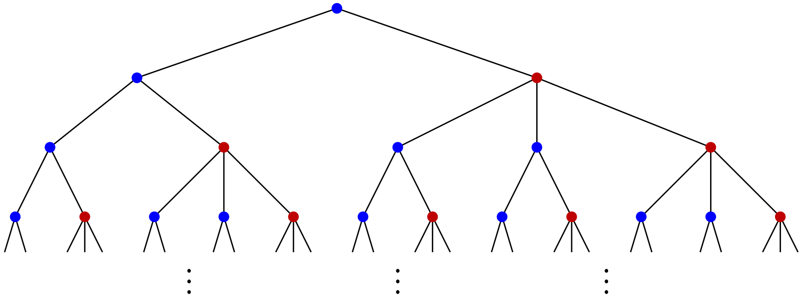
This is the ***tree of atoms***. Its space of ends is $\partial_h G$.

Hyperbolic Groups

Theorem (B–Bleak–Matucci 2018)

If G is a hyperbolic group, then:

1. *The tree of atoms has a self-similar structure, and*
2. *G acts on $\partial_h G$ by asynchronous automata.*



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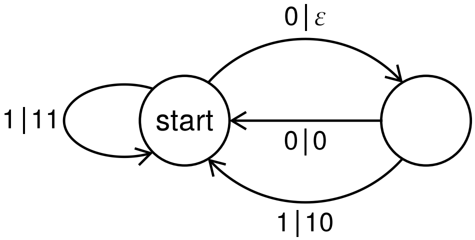

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Theorem (B–Bleak–Matucci–Zaremsky 2022)

The action of G on $\partial_h G$ is contracting, and hence $V[G]$ is finitely presented.

Open Questions

Which of the following groups embed into finitely presented simple groups?

1. Braid groups B_n for $n \geq 4$?
2. Mapping class groups?
3. $\mathrm{Out}(F_n)$?
4. Finitely generated nilpotent groups?
5. Finitely generated metabelian groups?
6. One relator groups?

Also, what is an explicit, natural example of a finitely presented group that contains $\mathrm{GL}_n(\mathbb{Q})$?

The End

Mapping Class Groups

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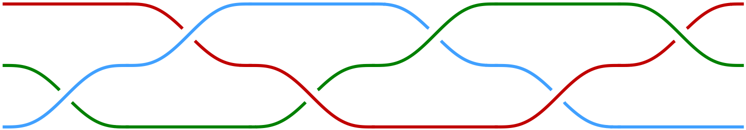
Open Question: Do mapping class groups and braid groups embed into finitely presented simple groups?



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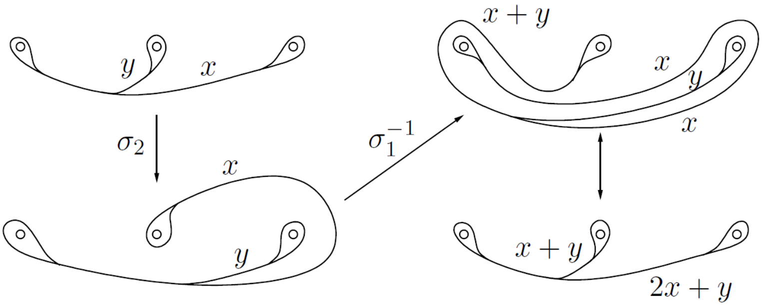

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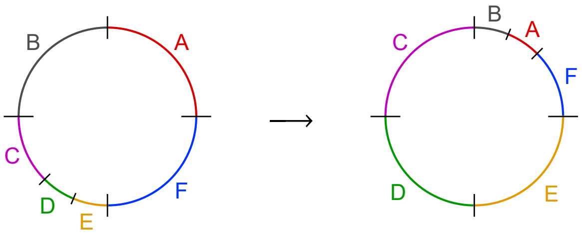

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If PIP(S^n) is finitely presented and simple, this would give Boone–Higman embeddings for all mapping class groups.