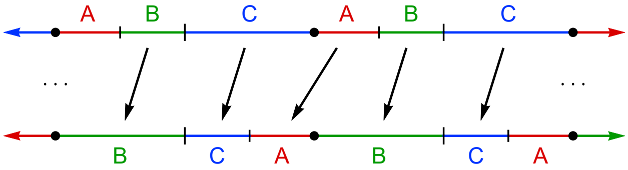
Embeddings into Finitely Presented Simple Groups



Jim Belk, University of Glasgow

An afternoon on group theory

ENS, 20 December 2022

The Boone–Higman Conjecture

The Boone–Higman Conjecture (1973)

Let G be a finitely generated group. Then:

G has solvable word problem	$\Leftrightarrow$	*G embeds into a finitely presented simple group*

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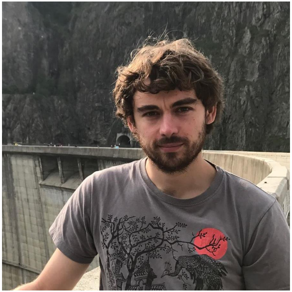
Recent progress: Many groups of interest embed into finitely presented simple groups.

Collaborators

Collin Bleak

University of St Andrews



James Hyde

University of Copenhagen

Collaborators

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Higman's Embedding Theorem

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A countable group presentation

$$\langle s_1, s_2, s_3, \ldots \mid r_1, r_2, r_3, \ldots \rangle$$

is ***computable*** if there exists an algorithm that outputs the list of relations.

A group is ***computably presented*** if it admits such a presentation.

Examples

1. Any finitely presented group.
2. Any finitely generated subgroup of a finitely presented group.

Higman's Embedding Theorem (1961)

Let G be a finitely generated group. Then:

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Graham Higman, 1960

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Corollary

Every countable abelian group embeds into a finitely presented group.

Follows from Higman–Neumann–Neumann 1949.

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Problem (Higman): Find an explicit and natural example of a finitely presented group that contains $\mathbb{Q}$.

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This theorem has the form

G has a certain algorithmic property	$\Leftrightarrow$	G embeds into a certain kind of group

Question (Higman): Are there other theorems of this type?

The Boone–Higman Conjecture

An Observation

Observation (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

Richard J. Thompson, 2004

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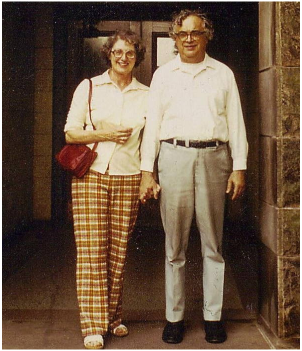
Thompson mentioned this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.



William and
Eileen Boone, 1979

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Proof.

Given a presentation $\langle s_1, \ldots s_m \mid r_1, \ldots r_n \rangle$ for a simple group G and a word w, we run two simultaneous searches:

Search #1	**Search #2**
Search for a proof that $$w = 1$$ using the relations $r_1, \ldots, r_n$.	Search for a proof that $$s_1 = \cdots = s_m = 1$$ using $w = 1$ and $r_1, \ldots, r_n$.

Eventually one of the searches terminates. □

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Boone and Higman recognized Thompson's observation as a group-theoretic analog of a basic observation in logic:

Observation

Every complete theory with finitely many axioms is decidable.

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Logic	**Group Theory**
axiomatic system	group presentation
axioms	relations
inconsistent theory	trivial group
complete theory	simple group
decidable theory	decidable word problem

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As a corollary, any countable abelian group would embed into a finitely presented simple group.

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Trick: Given words $u, v \neq_G 1$, consider the group

$$G' = \left\langle G, x, t \,\middle|\, (uu^x)^t = u^x v \right\rangle.$$

G' is an HNN extension of $G * \langle x \rangle$, so G embeds into G'.

But now v lies in the normal closure of u.

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Theorem (Sacerdote 1977)

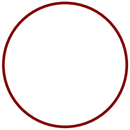
There are analogs of the Boone–Higman theorem for the order, conjugacy, power, and subgroup membership problems.

Finitely Presented Simple Groups

Thompson’s Groups

In 1965, Richard J. Thompson defined three infinite groups.

F acts on the interval.
finitely presented



T acts on the circle.
finitely presented, simple

V acts on the Cantor set.
finitely presented, simple

Definition of V

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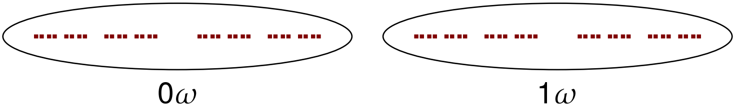

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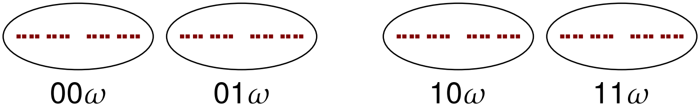

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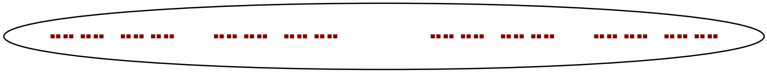

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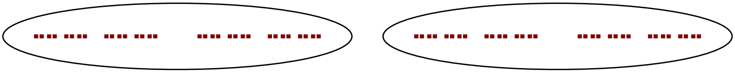

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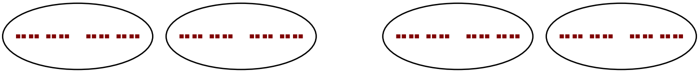

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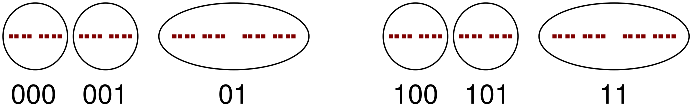

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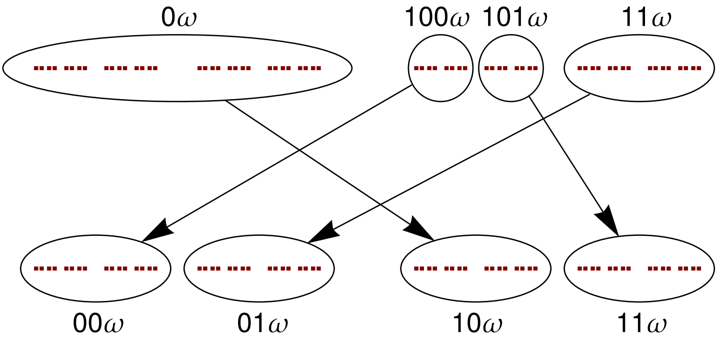
Definition of V

Thompson's group V is the group of all homeomorphisms that map "linearly" between the pieces of two dyadic subdivisions.



This group V is finitely presented and simple.

Subgroups of V

The following groups embed into V:

1. All finite groups, free groups, free abelian groups, $\bigoplus_{\omega} V$.
2. (Higman 1974, Brown 1987) Generalized Thompson groups F_n, T_n, and V_n.
3. (Röver 1999) The Houghton groups H_n, and free products of finitely many finite groups.
4. (Guba–Sapir 1999) $\mathbb{Z} \wr \mathbb{Z},\ (\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z},\ ((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \mathbb{Z},\ \ldots$
5. (Bleak–Kassabov–Matucci 2011) $\mathbb{Q}/\mathbb{Z}$.
6. (Bleak–Salazar-Díaz 2013) $V \wr A$ and $V * A$, where A is any finite group or $A \in \{\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\}$.

Grigorchuk’s Group

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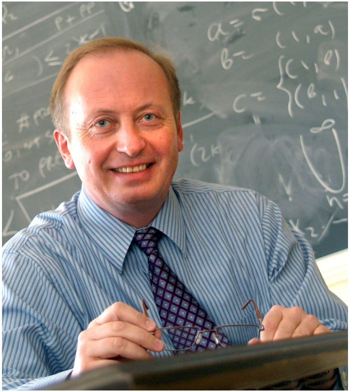
Grigorchuk (1979) defined a certain finitely generated group $\mathcal{G} \leq \mathrm{Aut}(T_2)$.



Rostislav Grigorchuk

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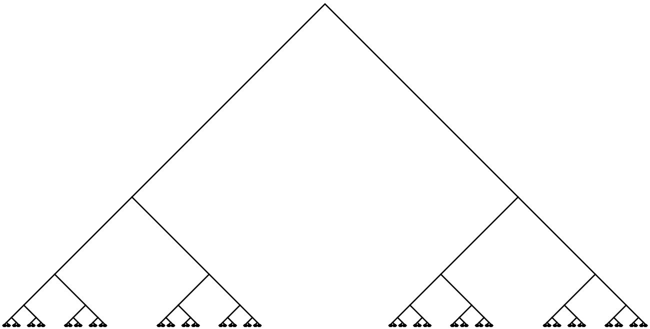

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Properties of $\mathcal{G}$ (Grigorchuk 1979 and 1984)

- $\mathcal{G}$ has intermediate growth: the number of elements of length less than n grows like $\exp\left(n^{0.7675}\right)$ (Erschler–Zheng 2020).
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Theorem (Röver 1999)

Every finitely generated torsion subgroup of V is finite. Hence $\mathcal{G}$ does not embed into V.

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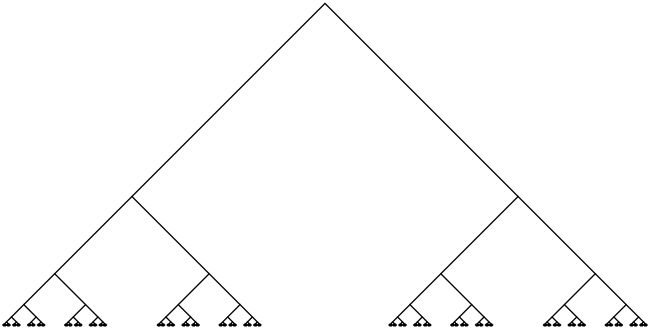
Observe that the boundary ∂T_2 is the Cantor set $\{0, 1\}^\omega$.



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Generalization: $V_d G$ is finitely presented whenever $G \leq \mathrm{Aut}(T_d)$ is self-similar and contracting (Nekrashevych 2013).

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But $V_d G$ is usually not simple.

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Matthew Brin

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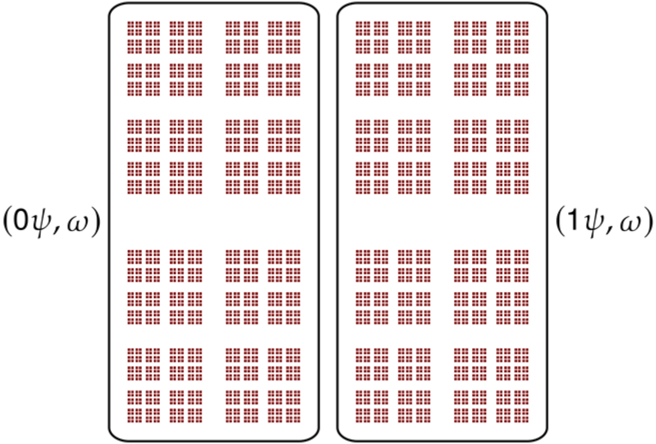

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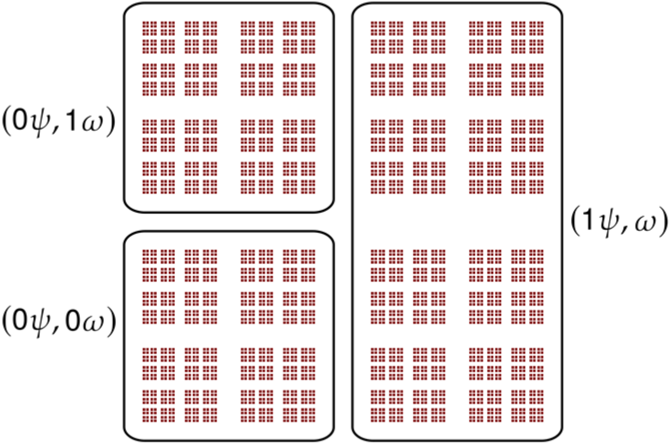

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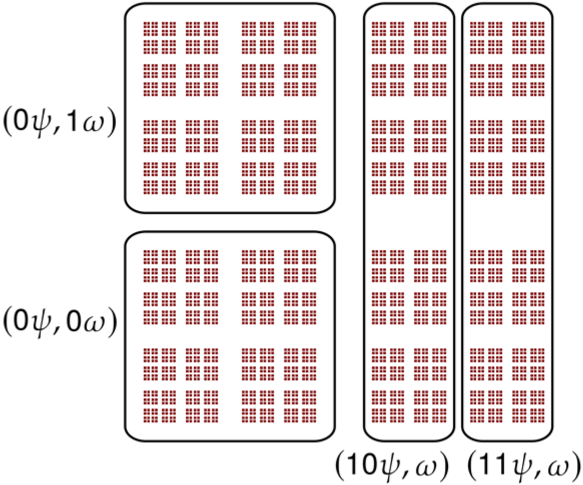

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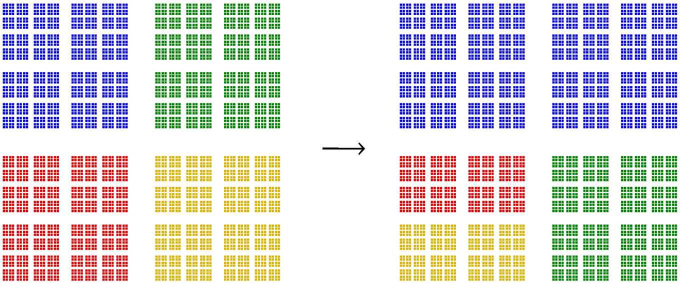
Elements of $2V$ map "linearly" between two subdivisions.



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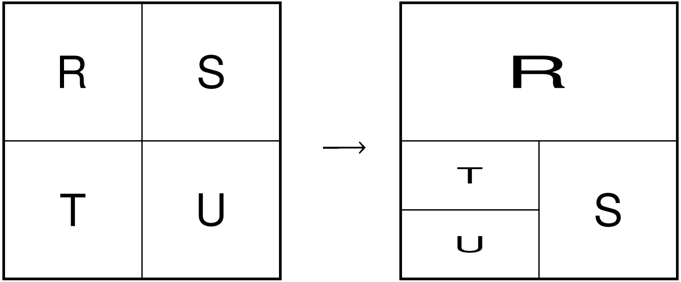

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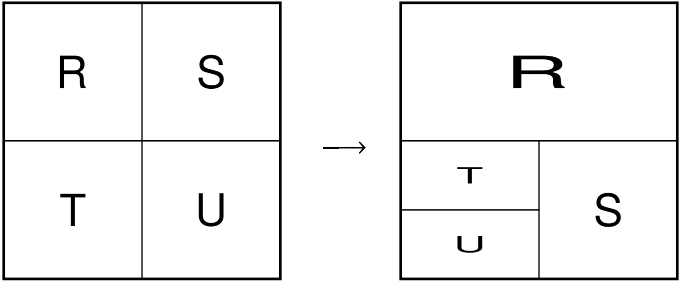


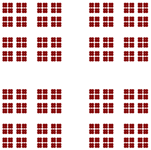
Theorem (Brin 2004)

The group $2V$ is finitely presented and simple.

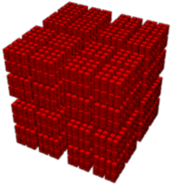
Brin's Groups

Brin defined a family of groups nV ($n \geq 1$) similarly, with $1V = V$.

V



$2V$



$3V$

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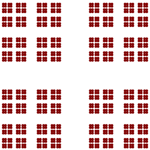

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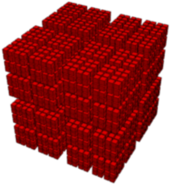

$3V$

$\cdots$

Theorem (Brin 2009)

The group nV is finitely presented and simple for all $n \geq 1$.

Algorithmic Properties

Brin's groups have very interesting algorithmic properties.

Theorem (B–Bleak 2014)

The order problem in nV is unsolvable for $n \geq 2$

Theorem (B–Bleak–Matucci 2016)

The subgroup membership problem in nV is unsolvable for $n \geq 2$.

Theorem (Salo 2020)

The conjugacy problem in nV is unsolvable for $n \geq 2$.

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Salo 2021: In fact, all RAAG's embed into $2V$.

Corollary

Any group G that virtually embeds into a RAAG embeds into $2V$.

Subgroups of Brin's groups

The following groups virtually embed into RAAG's, and therefore embed into $2V$:

1. (Crisp–Wiest 2004) All graph braid groups.
2. (Wise 2009) All limit groups.
3. (Haglund–Wise 2010) All finitely generated Coxeter groups.
4. (Agol 2012) All cubulated hyperbolic groups.
5. (Przytycki–Wise 2012) Fundamental groups of Riemannian 3-manifolds of non-positive curvature.
6. (Groves–Manning 2020, Oregón-Reyes 2020) Certain cubulated relatively hyperbolic groups.

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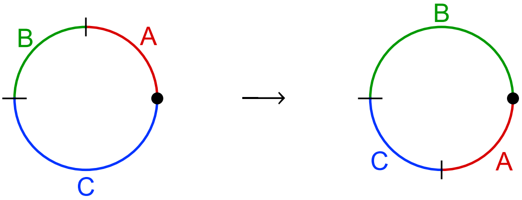
Theorem (B–Hyde–Matucci 2020)

The "lift" of Thompson's group T to the real line contains $\mathbb{Q}$.

$$\overline{T} = \langle a, b \mid a^4 b^{-3}, (ba)^5 b^{-9}, [bab, a^2baba^2], \\ [bab, a^2b^2a^2baba^2ba^2] \rangle$$

Countable Abelian Groups

Here ***Thompson's group T*** is a certain subgroup of Homeo(S^1).



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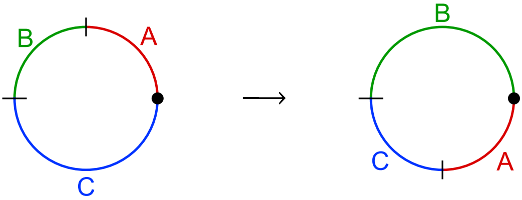
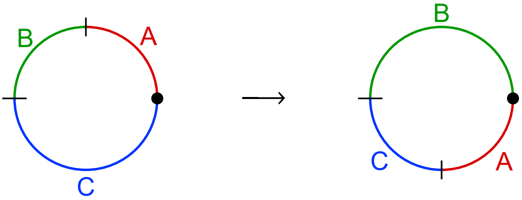


A ***lift*** of an element $g \in T$ is a homeomorphism $\overline{g} \colon \mathbb{R} \to \mathbb{R}$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\overline{g}} & \mathbb{R} \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow[g]{} & S^1 \end{array}$$

Countable Abelian Groups

Here ***Thompson's group T*** is a certain subgroup of Homeo(S^1).



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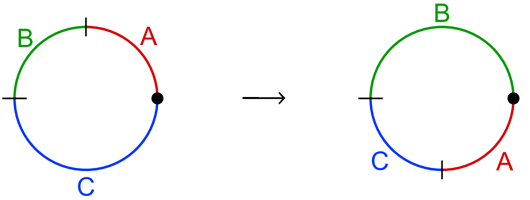
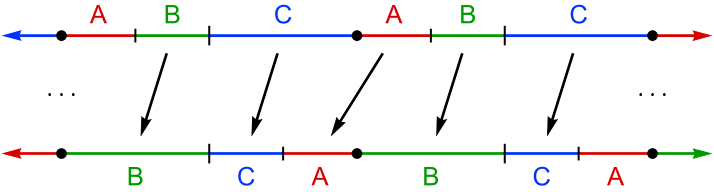


Here's one possible lift of the element above:



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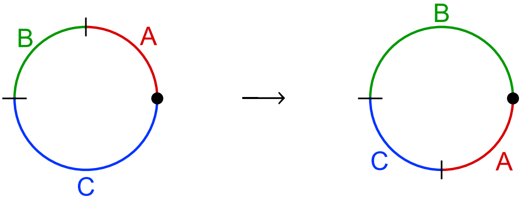

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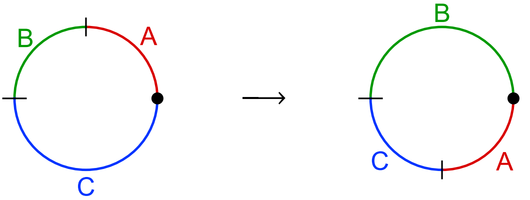

The group $\overline{T}$ (first defined in Ghys–Sergiescu 1980) consists of all lifts of all elements of T.

Bleak–Kassabov–Matucci 2011: T contains $\mathbb{Q}/\mathbb{Z}$.

B–Hyde–Matucci 2020: $\overline{T}$ contains $\mathbb{Q}$.

Countable Abelian Groups

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Proof Outline.

1. Define an action of $\overline{T}$ on the Cantor set $\{0,1\}^{\omega}$.
2. Prove that the group $V\overline{T}$ generated by V and $\overline{T}$ is finitely presented and simple.
3. Prove that $V\overline{T}$ contains $\bigoplus_{\omega}\mathbb{Q} \oplus \bigoplus_{\omega}\mathbb{Q}/\mathbb{Z}$, and thus contains every countable abelian group. □

Twisting Brin’s Groups

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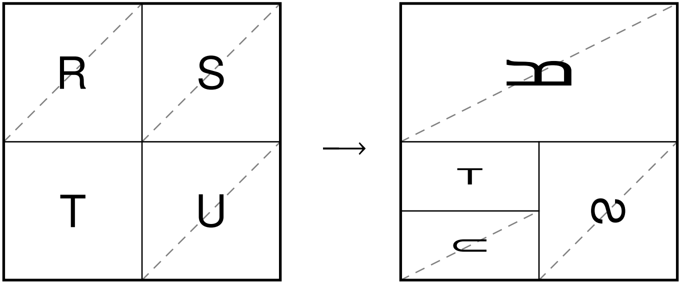

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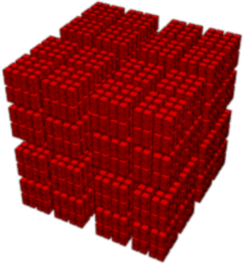
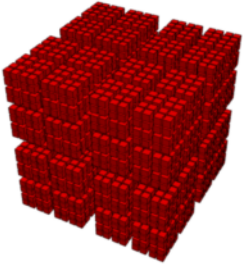


$\longrightarrow$



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Corollary (B–Zaremsky 2020)

Any finitely generated group G embeds isometrically into a finitely generated simple group.

Finite Presentation

Theorem (B–Zaremsky 2020, Zaremsky 2022)

Suppose:

1. *G is finitely presented,*
2. *G acts highly transitively on a set X, and*
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Corollary

Every contracting self-similar group G embeds into a finitely presented simple group.

Proof: G embeds into $V_d G$, and $V_d G$ satisfies the conditions above.

Hyperbolic Groups

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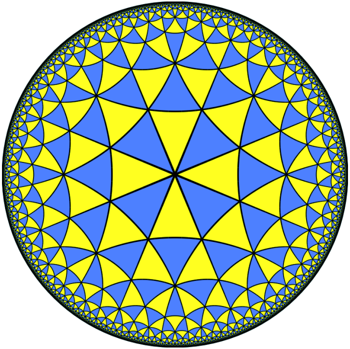
Theorem (B–Bleak–Matucci–Zaremsky 2022)

Every hyperbolic group G embeds into a finitely presented simple group.



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3. Prove that $V[G]$ is finitely presented, highly transitive on some orbit, and has finitely generated stabilizers, and hence the resulting twisted ωV is finitely presented.

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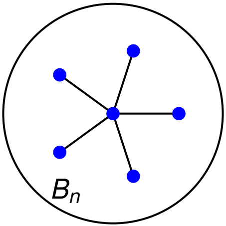

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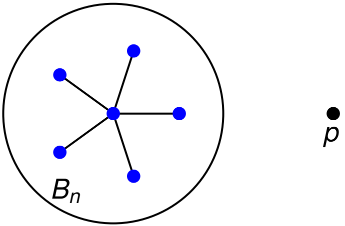

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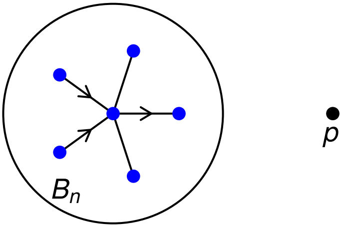

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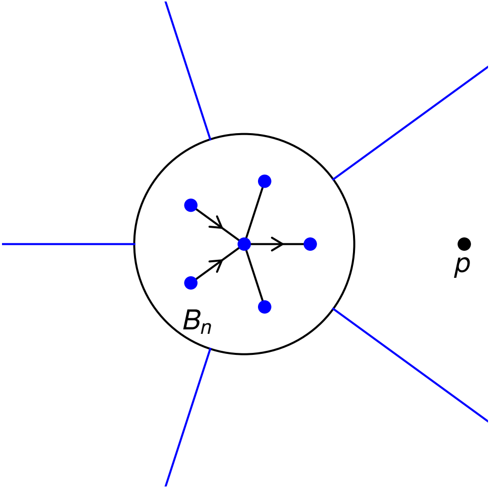

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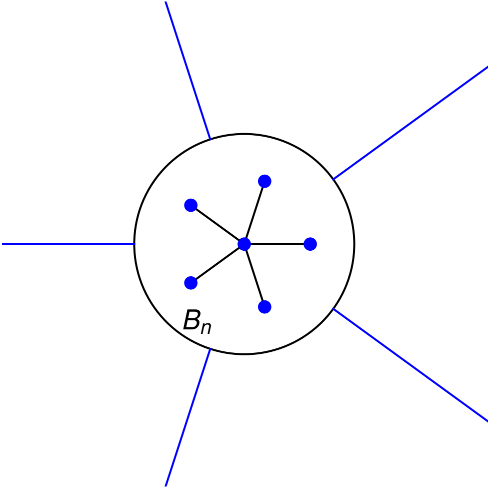

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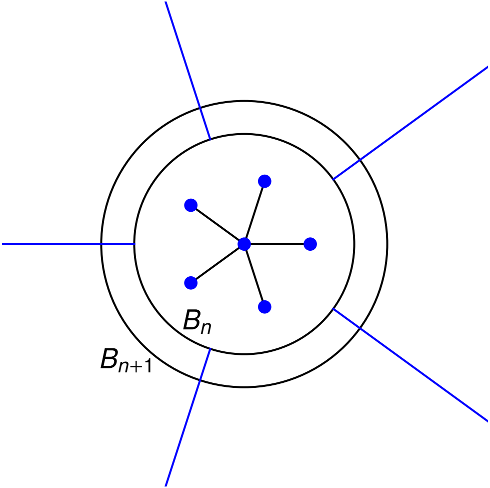

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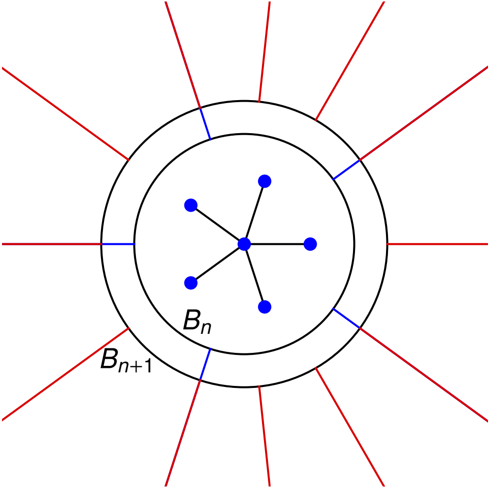

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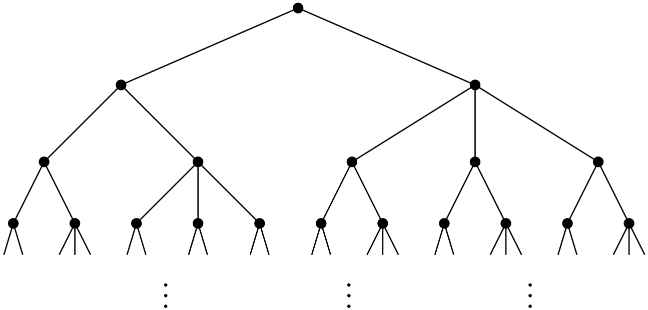


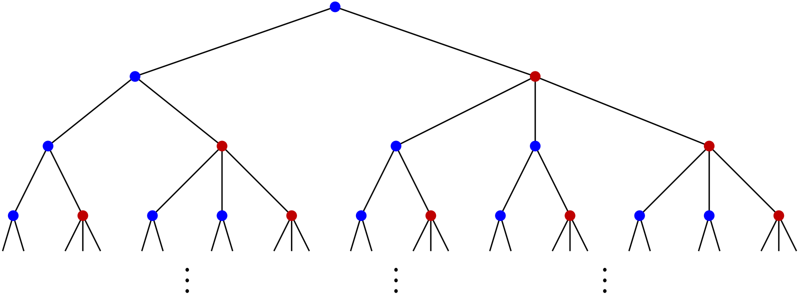
This is the ***tree of atoms***. Its space of ends is $\partial_h G$.

Hyperbolic Groups

Theorem (B–Bleak–Matucci 2018)

If G is a hyperbolic group, then:

1. *The tree of atoms has a self-similar structure, and*
2. *G acts on $\partial_h G$ by asynchronous automata.*



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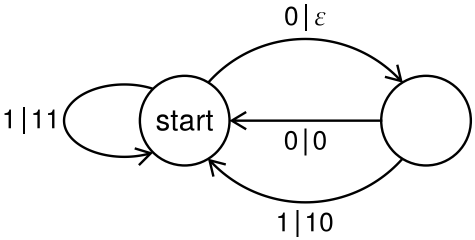

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Theorem (B–Bleak–Matucci–Zaremsky 2022)

The action of G on $\partial_h G$ is contracting, and hence $V[G]$ is finitely presented.

In particular, you always arrive at a state in the nucleus after at most $2|g| + 39\delta + 13$ steps.

Open Questions

Which of the following groups embed into finitely presented simple groups?

1. Braid groups B_n for $n \geq 4$?
2. Mapping class groups?
3. $\mathrm{Out}(F_n)$?
4. Finitely generated nilpotent groups?
5. Finitely generated metabelian groups?
6. One relator groups?

Also, what is an explicit, natural example of a finitely presented group that contains $\mathrm{GL}_n(\mathbb{Q})$?

Mapping Class Groups

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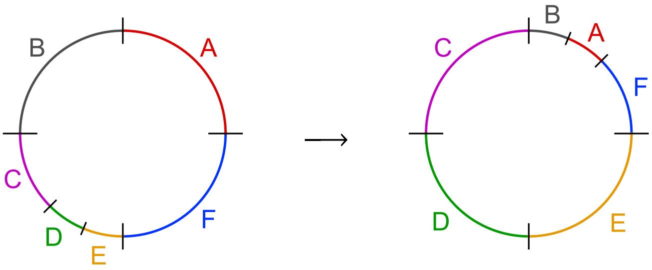
It is an open question whether mapping class groups and braid groups embed into finitely presented simple groups.

For one possible approach, consider the following element of Thompson's group T.



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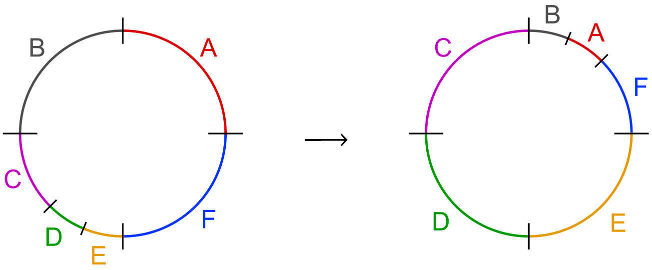

This looks just like the action of a pseudo-Anosov on $\mathcal{PMF}$!

Mapping Class Groups

Train tracks give $\mathcal{PMF}$ a ***piecewise-integral projective (PIP) structure***, with elements of $\mathrm{Mod}(S)$ acting as PIP maps.

Thurston observed that the group $\mathrm{PIP}(S^1)$ of PIP homeomorphisms of S^1 is isomorphic to Thompson's group T.

Open Question (Thurston): For $n \geq 2$, is the group $\mathrm{PIP}(S^n)$ finitely generated?

$\mathrm{Mod}(S_{g,n})$ embeds into $\mathrm{PIP}(S^{6g-7+2n})$ for $g \geq 3$. Is this a finitely presented simple group?

The End