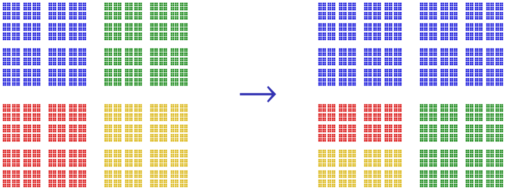
Boone–Higman Embeddings of $\mathrm{Aut}(F_n)$ and Mapping Class Groups of Punctured Surfaces



Jim Belk, University of Glasgow

North British Geometric Group Theory Seminar

University of St Andrews, 9 May 2025

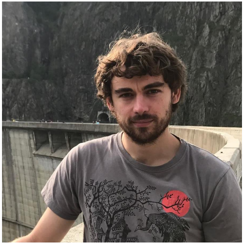
Collaborators

Collin Bleak

St Andrews

F. Fournier-Facio

Cambridge



James Hyde

SUNY Binghamton

Collaborators

Francesco Matucci

Milano–Bicocca

Matthew Zaremsky

SUNY Albany

The Word Problem

Let G be a group with finite generating set $S = \{s_1, \ldots, s_r\}$.

The Word Problem in G (Dehn 1911)

Given a word w in $s_1, \ldots, s_r$, decide whether w represents the identity in G.

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Theorem (Novikov 1955, Boone 1954–1957)

There exist finitely presented groups with unsolvable word problem.

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If W is the set of all words for the identity in G, then:

1. W is computably enumerable $\Leftrightarrow$ G is computably presented
2. W is computable $\Leftrightarrow$ G has solvable word problem

Theorem and conjecture

Higman's Embedding Theorem (1961)

Let G be a finitely generated group. Then:

G is computably presented	$\Leftrightarrow$	*G embeds into a finitely presented group*

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Easy Direction

Proposition (Kuznecov 1958, Thompson 1969)

Every finitely presented simple group has solvable word problem.

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Proof.

Given a presentation $\langle s_1, \dots s_m \mid R_1, \dots R_n \rangle$ for a simple group G and a word w, we run two simultaneous searches:

Search #1	Search #2
Search for a proof that $w = 1$ using the relations $R_1, \dots, R_n$.	Search for a proof that $s_1 = \cdots = s_m = 1$ using $w = 1$ and $R_1, \dots, R_n$.

Eventually one of the searches terminates. $\square$

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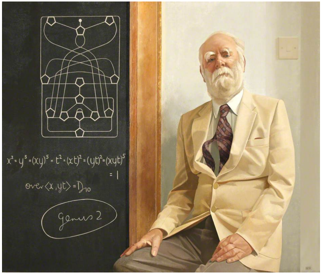
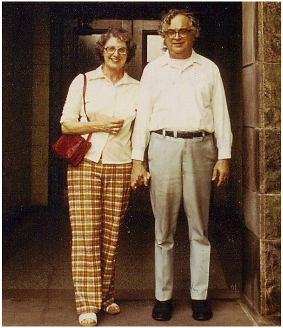
Thompson showed this result at a 1969 conference in Irvine, California. Higman and William Boone were both in the audience.



Graham Higman, 1984



William and Eileen Boone, 1979

Motivation

They immediately recognized this proof as a group-theoretic analog of a basic observation in logic:

Observation

Every complete theory with finitely many axioms is decidable.

Logic	**Group Theory**
axiomatic system	group presentation
axioms	relations
inconsistent theory	trivial group
complete theory	simple group
decidable theory	decidable word problem

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Let G be a finitely generated group. Then:

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Candidates: Free groups, $\mathrm{GL}(n, \mathbb{Z})$, hyperbolic groups, Coxeter groups, braid groups, mapping class groups, $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$, RAAGs, Baumslag–Solitar groups, lamplighter groups, many Artin groups, 3-manifold groups, polycyclic groups, ...

Finitely Presented Simple Groups

Thompson's Group V

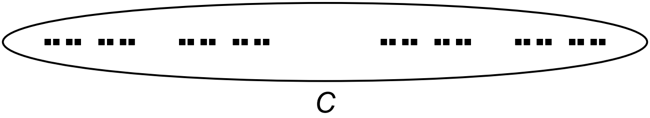
The first examples of infinite, finitely presented simple groups were introduced by Richard J. Thompson in the 1960's. The largest of these is ***Thompson's group V***.

Richard J. Thompson, 2004

Thompson's Group V

Let C be the Cantor space $\{0,1\}^\omega$.

Each finite binary sequence α determines a ***cone*** αC.



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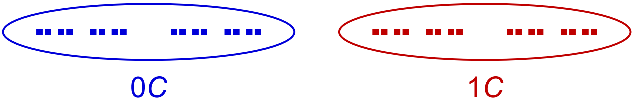

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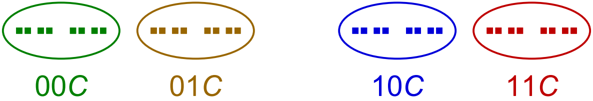

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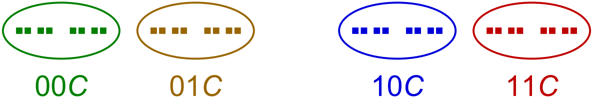
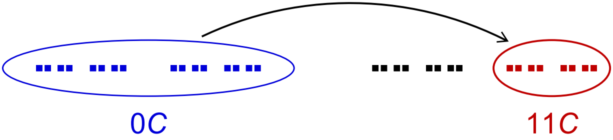
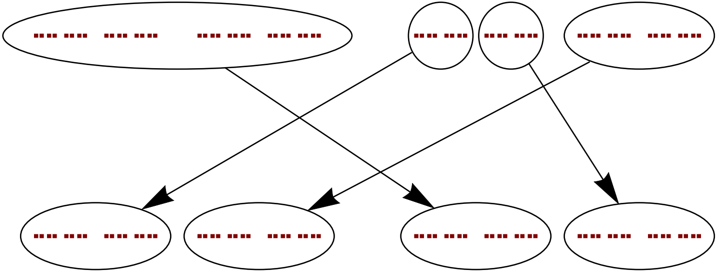


There is a ***prefix-replacement*** homeomorphism between any two cones.



Thompson's Group V

Elements of ***Thompson's group V*** map the cones of one partition to the cones of another by prefix replacement.



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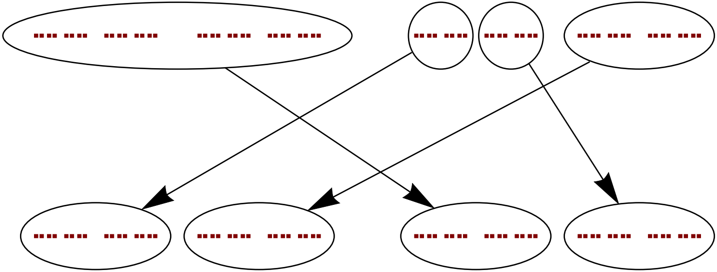
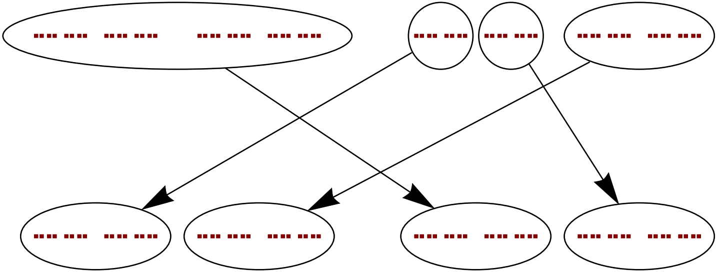


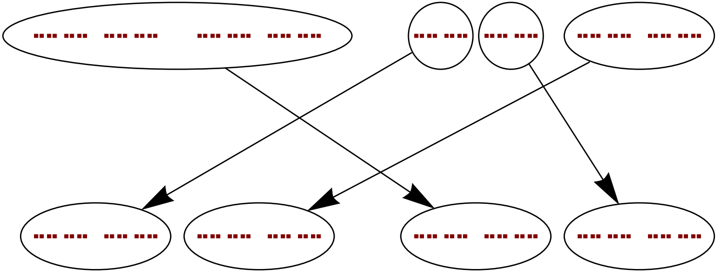
(Thompson 1965, Higman 1974) V is finitely presented and simple.

Properties of V



Properties of V

1. It acts on a ***Cantor space***, in this case the Cantor set $\{0,1\}^\omega$.



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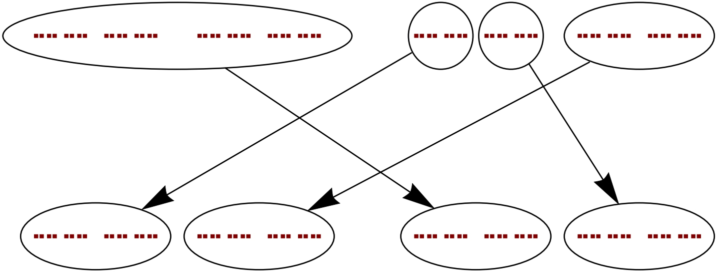
2. Elements are ***piecewise defined***. Every homeomorphism whose pieces have the right form belongs to the group.



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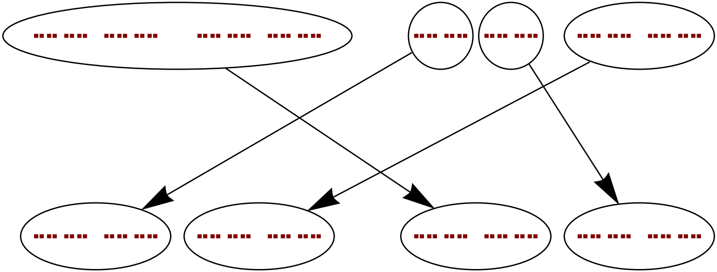

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Most known finitely presented simple groups have these properties, and are in some sense generalizations of V.

Brin–Thompson Groups

For example, Brin (2004) defined a group $2V$ acting on $C \times C$.

Matthew Brin

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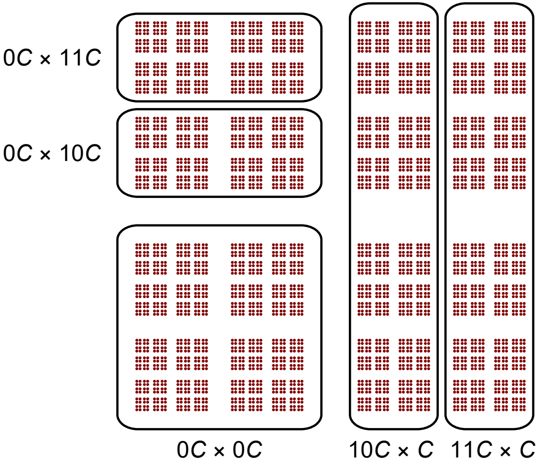
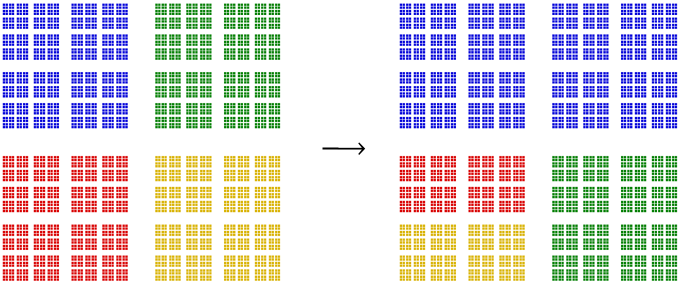


Brin–Thompson Groups

Elements of $2V$ map "linearly" between two subdivisions.



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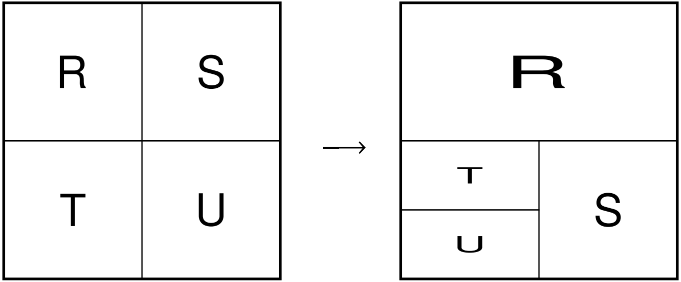

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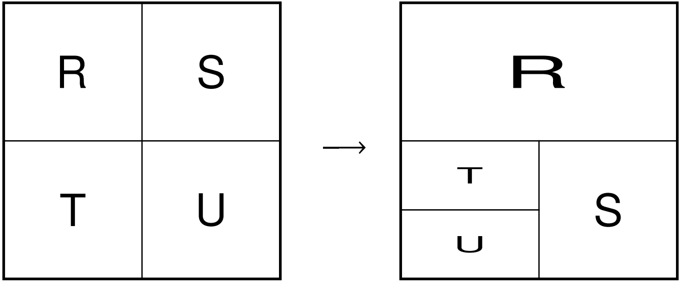
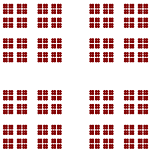


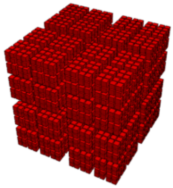
Theorem (Brin 2004)

The group $2V$ is finitely presented and simple.

Brin–Thompson Groups

Brin defined a family of groups nV ($n \geq 1$) similarly, with $1V = V$.





$\ldots$

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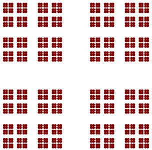

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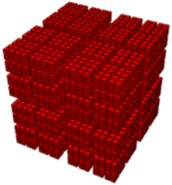


$3V$

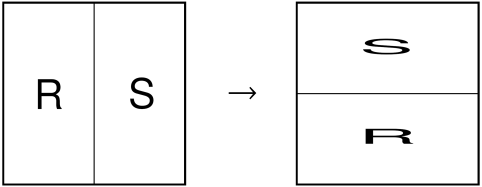
$\cdots$

Theorem (Brin 2009)

The group nV is finitely presented and simple for all $n \geq 1$.

Properties of Brin–Thompson groups

The Brin–Thompson groups have very interesting dynamics.



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The Brin–Thompson groups have very interesting dynamics.

They also have very interesting algorithmic properties.

Theorem (B–Bleak 2014)

The torsion problem in nV is unsolvable for $n \geq 2$

Theorem (B–Bleak–Matucci 2016)

The subgroup membership problem in nV is unsolvable for $n \geq 2$.

Some Boone–Higman Embeddings

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Virtually free groups (Röver 1999) and various wreath products (Guba–Sapir 1999, Bleak–Salazar-Díaz 2013) also embed into V, but not much else.

Scott's groups

Scott (1984) found finitely presented simple groups $\mathrm{Sc}(n)$ that contain $\mathrm{GL}_n(\mathbb{Z})$.

Elizabeth Scott

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Observe that $\mathrm{GL}_n(\mathbb{Z})$ acts on the Cantor space $\mathbb{Z}_{(2)}^n$, where $\mathbb{Z}_{(2)}$ is the 2-adic integers.

$\mathrm{Sc}(n)$ is the group of all piecewise homeomorphisms f of $\mathbb{Z}_{(2)}^n$ whose pieces have the form

$$f(\mathbf{x}) = 2^r A\mathbf{x} + 2^s\mathbf{b} \qquad r, s \in \mathbb{Z},\ A \in \mathrm{GL}_n(\mathbb{Z}),\ \mathbf{b} \in \mathbb{Z}^n$$

Theorem (Scott 1984)

The group $\mathrm{Sc}(n)$ is finitely presented, simple, and contains $\mathrm{GL}_n(\mathbb{Z})$.

Consequences

The following finitely presented groups embed into $\mathrm{GL}_n(\mathbb{Z})$:

1. Auslander (1967) Virtually polycyclic groups (including virtually nilpotent groups)
2. Hsu–Wise (1999) Right-angled Artin groups
3. Crisp–Wiest (2004) Surface groups
4. Haglund–Wise (2010) Coxeter groups
5. Agol (2013) Cubulated hyperbolic groups
6. Przytycki–Wise (2018) Mixed 3-manifold groups
7. Wise (2021) Limit groups, one-relator groups with torsion

By Scott's theorem, all such groups admit Boone–Higman embeddings.

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3. Bux–Isenrich–Wu (2024): Baumslag–Solitar groups, free-by-cyclic groups, Artin groups of types $\widetilde{A}_2$, $\widetilde{C}_2$, and $\widetilde{G}_2$
4. B–Hyde–Matucci (2024): Countable abelian groups (e.g. $\mathbb{Q}$)
5. Zaremsky (2025): Finitely presented self-similar groups, S-arithmetic groups
6. B–Fournier-Facio–Hyde–Zaremsky (2025): $\mathrm{Aut}(F_n)$, braid groups, some mapping class groups, Artin groups of types B_n, D_n, $I_2(m)$, and $\widetilde{A}_n$

Type (A) Actions

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Let G be a finitely presented group acting faithfully on a set X.

The action $G \curvearrowright X$ has ***type* (A)** if:

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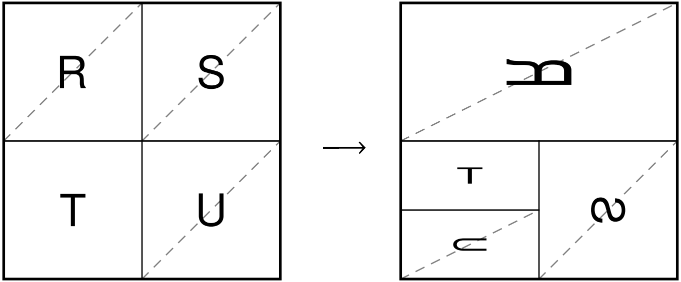
Theorem (B–Zaremsky 2020, Zaremsky 2024)

Any group that admits a type (A) *action has a Boone–Higman embedding.*

The proof uses a new class of simple groups called ***twisted Brin–Thompson groups***.

Twisted Brin–Thompson groups

For example, ***twisted* $2V$** is similar to $2V$, but we can reflect rectangles along their main diagonal.



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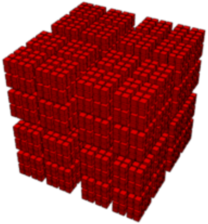
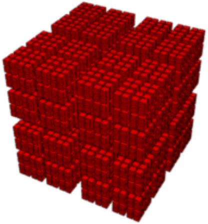
Any permutation group $G \curvearrowright \{1, \ldots, n\}$ determines a twisted version of nV.



$\longrightarrow$



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More generally, any faithful action $G \curvearrowright X$ gives you a ***twisted Brin–Thomspon group***.

Theorem (B–Zaremsky 2020)

Any twisted Brin–Thompson group is simple, and contains G.

Theorem (Zaremsky 2024)

A twisted Brin–Thompson group is finitely presented if and only if $G \curvearrowright X$ has type (A).

Example: Hyperbolic Groups

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Theorem (B–Bleak–Matucci–Zaremsky 2023)

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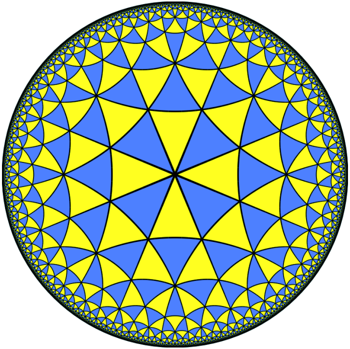

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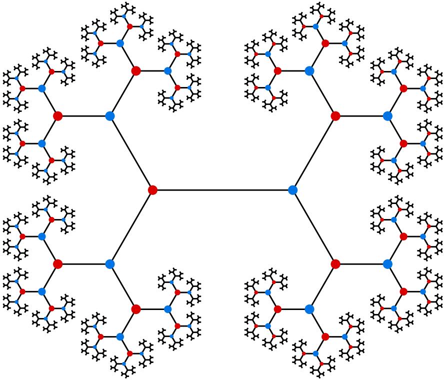
Theorem (Bux–Isenrich–Wu 2024)

Let G be a group acting faithfully and cocompactly on a locally finite tree T. If all vertex stabilizers are finitely presented, then G admits a Boone–Higman embedding.



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Sketch of Proof:

1. Define a group $\mathrm{RP}_G(T)$ of permutations of the vertices of T that contains G.
2. Prove that $\mathrm{RP}_G(T)$ is finitely presented and its action on the vertices has type (A).
3. So G embeds in $\mathrm{RP}_G(T)$, and $\mathrm{RP}_G(T)$ embeds in a finitely presented simple group.

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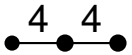
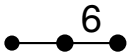
This theorem gives Boone–Higman embeddings for:

1. Bamuslag–Solitar groups $\langle a, b \mid b^{-1}a^m b = a^n \rangle$,
2. Free-by-cyclic groups $F_n \rtimes \mathbb{Z}$, and
3. Certain Artin groups:

$\widetilde{A}_2$: 

$\widetilde{C}_2$: 

$\widetilde{G}_2$: 

$\mathrm{Aut}(F_n)$

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Each $\mathrm{Aut}(F_n)$ *admits a Boone–Higman embedding into a twisted Brin–Thompson group.*

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Each Aut(F_n) *admits a Boone–Higman embedding into a twisted Brin–Thompson group.*

Corollary

The following groups admit Boone–Higman embeddings:

1. *The braid groups $\mathcal{B}_n$.*
2. *The loop braid groups $\mathcal{LB}_n$ and $\mathcal{LB}_n^{ext}$.*
3. *The ribbon braid groups $\mathcal{RB}_n$.*

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Corollary

Mapping class groups for the following surfaces admit Boone–Higman embeddings:

1. *Any finite-type surface with at least one puncture.*
2. *Any finite-type surface with at least one boundary component.*
3. *The closed surface of genus 2.*

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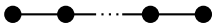
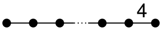
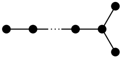
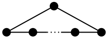
The following Artin groups admit Boone–Higman embeddings:

A_n : 

B_n : 

D_n : 

$I_2(m)$: 

$\widetilde{A}_n$: 

Finding an action

Let $F_n = \langle x_1, \ldots, x_n \rangle$.

Then $\mathrm{Aut}(F_n)$ is generated by ***Nielsen transformations***:

1. $(x_1, x_2, \ldots, x_n) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for all $\sigma \in S_n$
2. $(x_1, x_2, \ldots, x_n) \mapsto (x_1^{-1}, x_2, \ldots, x_n)$
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If G is any group, then $\mathrm{Aut}(F_n)$ acts on $G^n = G \times \cdots \times G$.

For an action of type (A), we need:

1. The action to be faithful,
2. The action to have finitely many orbits of pairs, and
3. The stabilizers to be finitely generated.

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Recall that a ***law*** in a group G is an equation in n variables that holds for all choices of elements of G.

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For example,

$$x_1 x_2 = x_2 x_1$$

is a 2-variable law that holds in any abelian group.

Faithfulness

Recall that a ***law*** in a group G is an equation in n variables that holds for all choices of elements of G.

For example,

$$x_1 x_2 x_1^{-1} x_2^{-1} = 1$$

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If $L = 1$ is a law in G, where L is a word in $x_1, \ldots, x_n$, then

$$(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, x_{n+1}L)$$

is an element of $\mathrm{Aut}(F_{n+1})$ that acts trivially on G^{n+1}.

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G is ***lawless*** if it has no nontrivial laws. For such a group, the action

$$\mathrm{Aut}(F_n) \curvearrowright G^n$$

is faithful for all n.

Orbits of pairs

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Unfortunately, the action

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Indeed, $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_n)$ can only be in the same orbit if

$$\langle g_1, \ldots, g_n \rangle = \langle h_1, \ldots h_n \rangle$$

so when G is infinite the action has infinitely many orbits.

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This makes a larger group $\langle \mathrm{Aut}(F_n), G \rangle$ that still acts on G^n.

Note: Elements of $\langle \mathrm{Aut}(F_n), G \rangle$ can be interpreted as automorphisms of $F_n * G$ that fix G pointwise.

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Theorem (B–Fournier-Facio–Hyde–Zaremsky 2025)

If G is simple and $n \geq 2$, then the action $\langle \mathrm{Aut}(F_n), G \rangle \curvearrowright G^n$ is 2-transitive.

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   If $g, h \in G$, then

   $$(x, y) \mapsto (xg, yh)$$

   is in $\langle \mathrm{Aut}(F_2), G\rangle$ and maps $(1, 1)$ to (g, h).

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   If $k_1, \ldots, k_m \in G$, then

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   If $k_1, \ldots, k_m \in G$, then

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   stabilizes $(1, 1)$. Since G is simple, we can use this to map any (g, h) to any (g', h) as long as $h \neq 1$. $\square$

Faithfulness

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For example, if $g \in Z(G)$, then

$$x_1 g x_1^{-1} g^{-1} = 1$$

is a 1-variable mixed identity in G.

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If $M = 1$ is a mixed identity in G, where M is a word in $g_1, \ldots, g_m, x_1, \ldots, x_n$ then

$$(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, x_{n+1}M)$$

is an element of $\langle \mathrm{Aut}(F_{n+1}), G \rangle$ that acts trivially on G^{n+1}.

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G is ***mixed identity free*** if it has no nontrivial mixed identities. For such a group, the action

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Theorem (Hull–Osin 2016)

Every countable, highly transitive group is either mixed identity free or has a normal subgroup isomorphic to $\mathrm{Alt}(\mathbb{N})$.

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In particular, Thompson's group V is mixed identity free.

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is faithful for all n.

So we're done

Theorem (B–Fournier-Facio–Hyde–Zaremsky)

For any $n \geq 2$,

1. *The action $\langle \mathrm{Aut}(F_n), V \rangle \curvearrowright V^n$ is faithful and 2-transitive,*
2. *The group $\langle \mathrm{Aut}(F_n), V \rangle$ is finitely presented, and*
3. *The stabilizer of any element of V^n is finitely generated,*

so we get a Boone–Higman embedding of $\langle \mathrm{Aut}(F_n), V \rangle$.

Notes:

- For (2), we use a theorem of Carette (2011).
- For (3), we prove that the stabilizer is a quasi-retract of $\langle \mathrm{Aut}(F_n), V \rangle$.

...and, there's more!

We can replace V by any highly transitive, finitely presented simple group. Thus:

Theorem (B–Fournier-Facio–Hyde–Zaremsky 2025)

Every highly transitive, finitely presented simple group embeds into a finitely presented twisted Brin–Thompson group.

...and, there's more!

We can replace V by any highly transitive, finitely presented simple group. Thus:

Theorem (B–Fournier-Facio–Hyde–Zaremsky 2025)

Every highly transitive, finitely presented simple group embeds into a finitely presented twisted Brin–Thompson group.

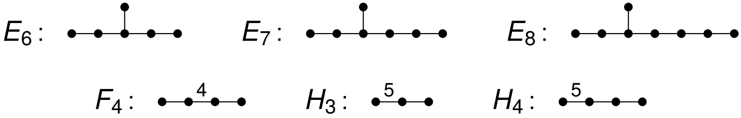
Permutational Boone–Higman Conjecture (Zaremsky 2024)

Every finitely presented group with solvable word problem embeds into a highly transitive, finitely presented simple group.

Open Questions

Which of the following Artin groups admit Boone–Higman embeddings?

- Exceptional groups of spherical type:



- Groups of Euclidean type other than $\widetilde{A}_n$, $\widetilde{C}_2$, and $\widetilde{G}_2$.

- Rank 3 groups of hyperbolic type? 3-free groups?

Open Questions

Which of the following groups admit Boone–Higman embeddings?

1. Mapping class groups for closed surfaces of genus ≥ 3
2. $\mathrm{Out}(F_n)$
3. $\mathrm{GL}_n(\mathbb{Q})$
4. Free Burnside groups $B(m, n)$ with solvable word problem
5. Finitely presented metabelian groups
6. One relator groups without torsion
7. CAT(0) groups
8. Automatic groups
9. Finitely presented residually finite groups

The End

Mapping Class Groups

It is an open question whether other mapping class groups admit Boone–Higman embeddings.

Train tracks give $\mathcal{PMF}$ a ***piecewise-integral projective (PIP) structure***, with mapping classes acting as PIP maps.

Thurston observed that the group $\mathrm{PIP}(S^1)$ of PIP homeomorphisms of S^1 is isomorphic to Thompson's group T.

Open Question (Thurston): For $n \geq 2$, is the group $\mathrm{PIP}(S^n)$ finitely generated?

If $\mathrm{PIP}(S^n)$ is a finitely presented simple group, this would give Boone–Higman embeddings for all mapping class groups.