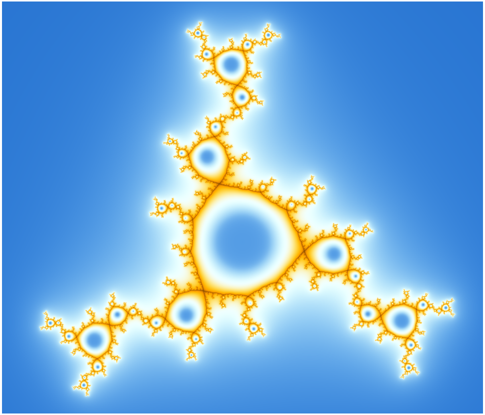
# Quasisymmetries of Finitely Ramified Julia Sets



Jim Belk, University of Glasgow

# Joint Work

Bradley Forrest

Stockton University

# Quasiconformal Geometry

# Quasiconformal Maps

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, let

$$\lfloor T \rfloor = \min_{v \neq 0} \frac{\|Tv\|}{\|v\|} \qquad \text{and} \qquad \lceil T \rceil = \max_{v \neq 0} \frac{\|Tv\|}{\|v\|}$$

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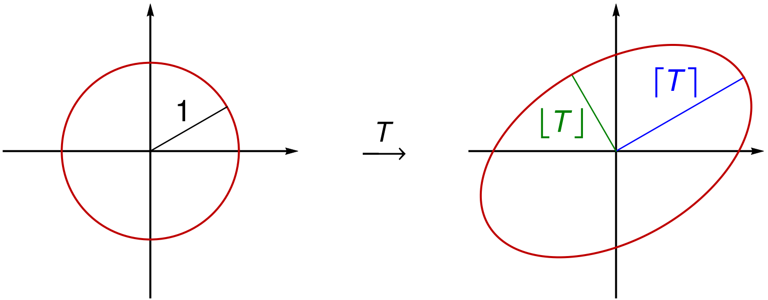


The ratio $\lceil T \rceil / \lfloor T \rfloor$ is a measure of ***eccentricity***.

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A diffeomorphism $f : U \longrightarrow U'$ between open subsets of $\mathbb{R}^n$ is ***quasiconformal*** if the function

$$p \longmapsto \frac{\lceil Df_p \rceil}{\lfloor Df_p \rfloor}$$

is bounded on $U$.

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For a linear transformation $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, let

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**Note:** If $\dfrac{\lceil Df_p \rceil}{\lfloor Df_p \rfloor} \equiv 1$ then $f$ is conformal (or anticonformal).

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**Note 2:** This definition can be extended to non-differentiable homeomorphisms.

# Applications of Quasiconformal Geometry

- ***Teichmüller theory:*** Defines a metric on the Teichmüller space of a hyperbolic surface (Teichmüller 1940). Leads to a proof of the Nielsen–Thurston classification of mapping classes (Bers 1978) and Thurston's theorem (Thurston 1982).

- ***No wandering domains:*** Every component of the Fatou set for a rational map is periodic or pre-periodic (Sullivan 1985).

- ***Mostow rigidity:*** For $n \geq 3$, if $X$ and $Y$ are closed hyperbolic $n$-manifolds and $\pi_1(X) \cong \pi_1(Y)$ then $X$ and $Y$ are isometric (Mostow 1968).

# Applications of Quasiconformal Geometry

- ***Geometric group theory:*** Any finitely generated group which is quasi-isometric to $\mathbb{H}^n$ has a geometric action on $\mathbb{H}^n$ (Tukia 1986, Gromov 1987, Cannon–Cooper 1992).

- ***Characteristic classes:*** Every $n$-manifold ($n \neq 4$) supports a unique quasiconformal structure (Sullivan 1978). This allows a theory of characteristic classes for such manifolds (Connes–Sullivan–Teleman 1994).

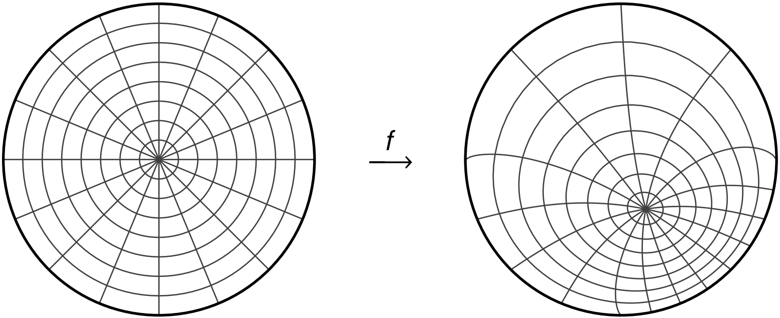
- ***Elliptic PDE's:*** Solution to Calderón's problem on electrical impedance tomography in two dimensions (Astala–Päivärinta 2006).

# Quasisymmetries

# Quasiconformal Maps on a Disk

Let $f : D^2 \longrightarrow D^2$ be a homeomorphism which is quasiconformal on the interior.

What can the restriction of $f$ to $S^1$ look like?



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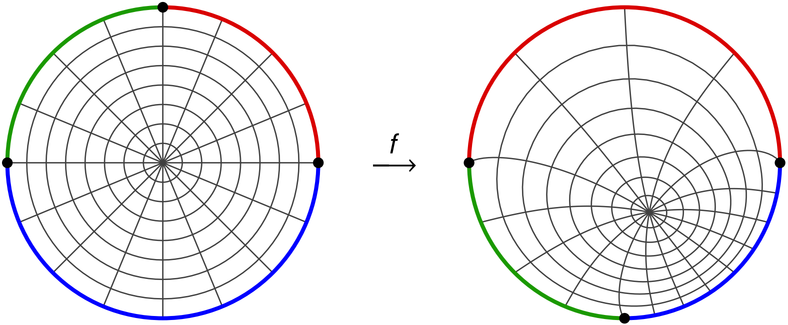

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### Theorem (Beurling–Ahlfors 1956)

*A homeomorphism $f\colon S^1 \longrightarrow S^1$ is a restriction of a quasiconformal map on $D^2$ iff there exists a homeomorphism $\eta\colon [0,\infty) \longrightarrow [0,\infty)$ so that*

$$\frac{\|f(a)-f(b)\|}{\|f(a)-f(c)\|} \leq \eta\left(\frac{\|a-b\|}{\|a-c\|}\right)$$

*for every triple $a, b, c$ of distinct points in $S^1$.*

# General Definition

Tukia and Väisälä (1980) observed that the Beurling–Ahlfors condition makes sense for homeomorphisms $f\colon X \longrightarrow Y$ between arbitrary metric spaces.

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A homeomorphism $f\colon X \longrightarrow Y$ is a ***quasisymmetry*** if there exists a homeomorphism $\eta\colon [0,\infty) \longrightarrow [0,\infty)$ such that

$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \leq \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

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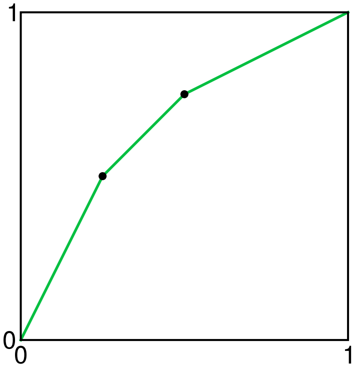
**Note:** The quasisymmetries $X \longrightarrow X$ form a group.

# Examples

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If $f$ is bilipschitz with

$$\frac{1}{K}\,d(x,x') \leq d\big(f(x),f(x')\big) \leq K\,d(x,x')$$

then $f$ is quasisymmetric with $\eta(t) = K^2 t$.

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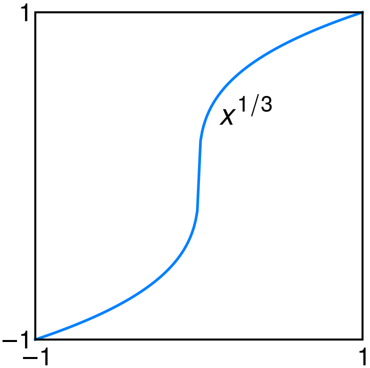


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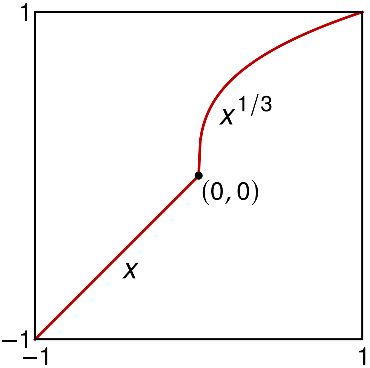
The function $f(x) = x^{1/3}$ is a quasisymmetry of $[-1,1]$, with

$$\eta(t) = \begin{cases} 6t^{1/3} & \text{if } 0 \leq t \leq 1 \\ 6t & \text{if } t > 1. \end{cases}$$

# A Non-Example




$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} \leq \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

This function is **not** a quasisymmetry of $[-1, 1]$.

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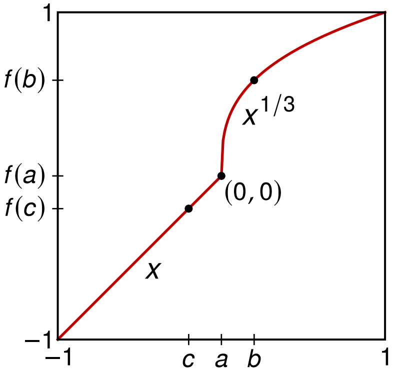


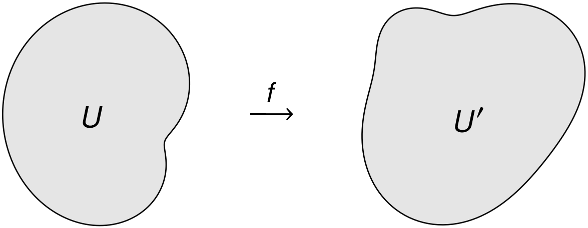

$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

For $a = 0$, $b = \varepsilon$, and $c = -\varepsilon$, we have

$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} = \frac{\varepsilon^{1/3}}{\varepsilon} = \frac{1}{\varepsilon^{2/3}} \qquad \text{and} \qquad \frac{d(a,b)}{d(a,c)} = 1.$$

# Quasiconformal vs. Quasisymmetric

Let $f : U \to U'$ be a homeomorphism between domains in $\mathbb{R}^n$.



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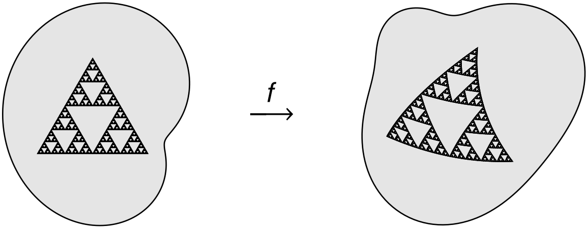


### Theorem (Väisälä 1981)

*If $f$ is quasiconformal then $f$ restricts to a quasisymmetry on every compact subset of $U$.*

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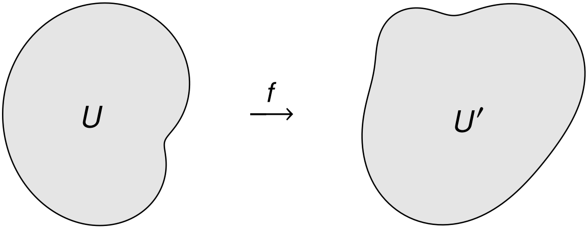

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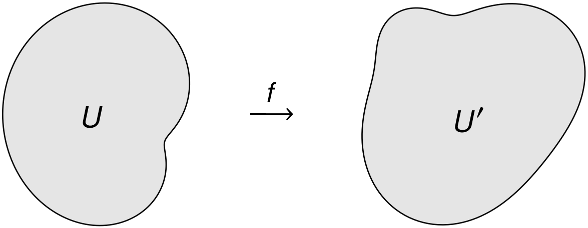


### Theorem (Egg Yolk Principle, Väisälä 1981)

*The following are equivalent:*

1. *$f$ is quasiconformal.*
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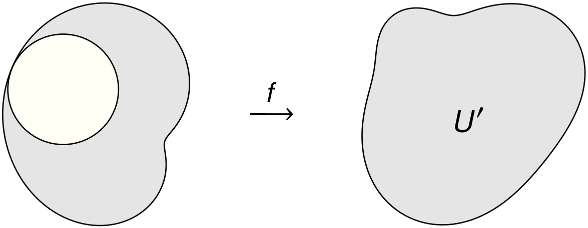


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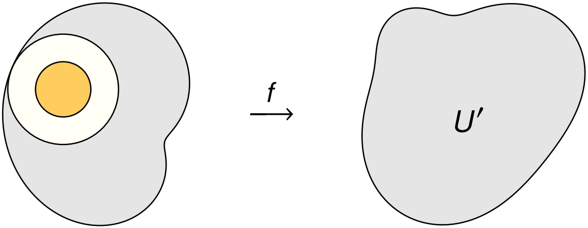


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# Relation to Hyperbolic Groups

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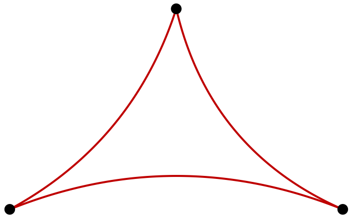

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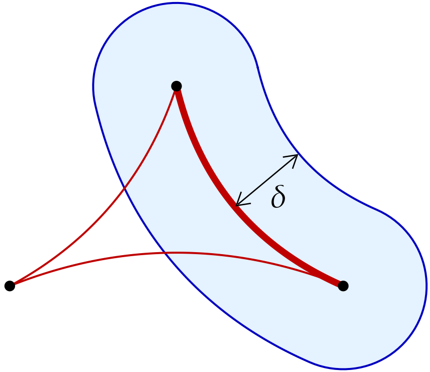

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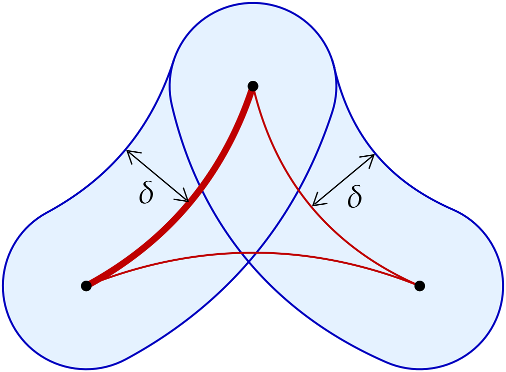

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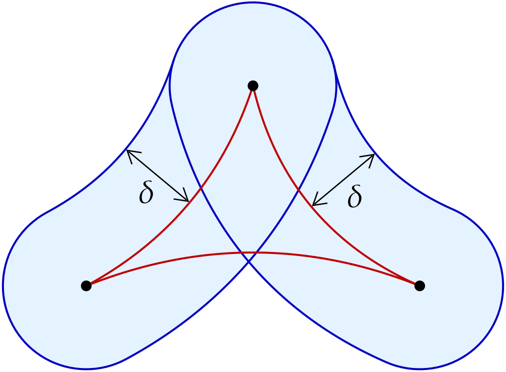

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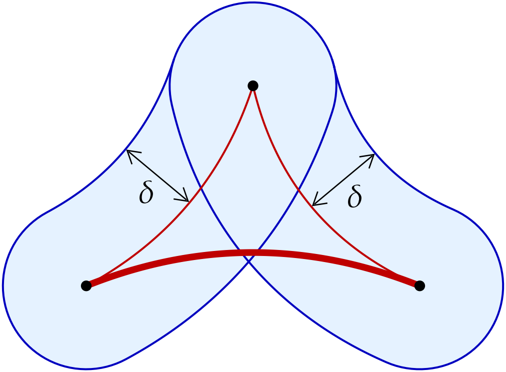

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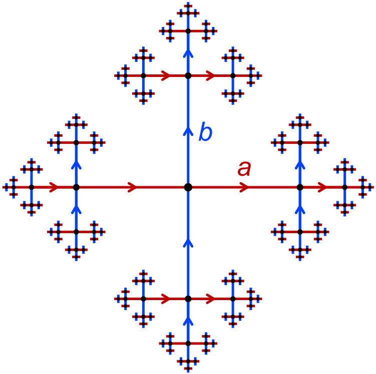

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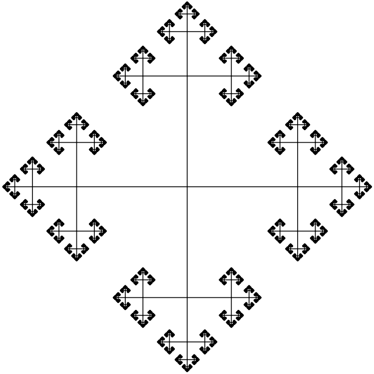

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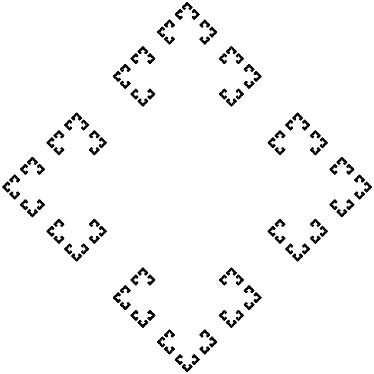

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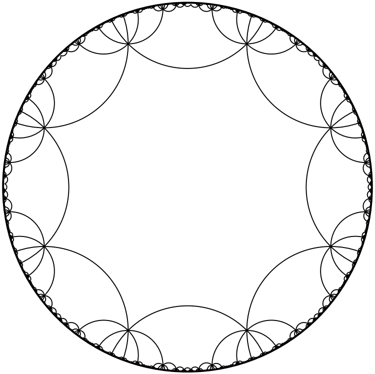

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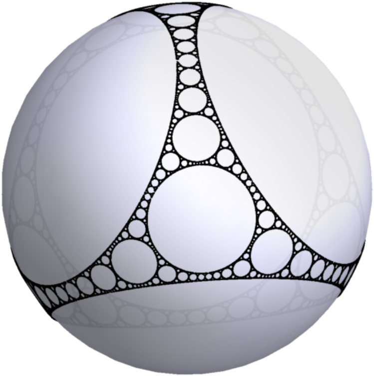

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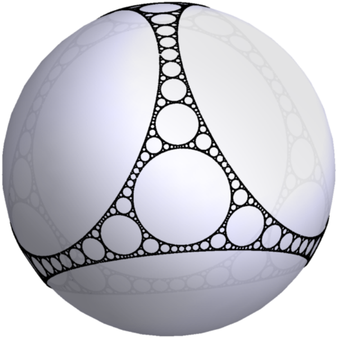

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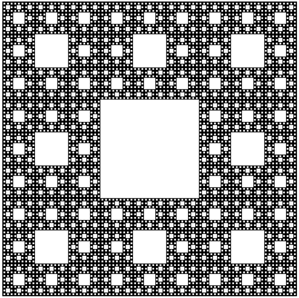
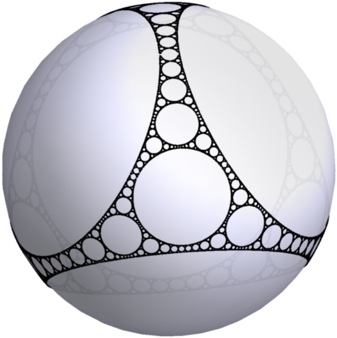


Sierpiński carpet

# Hyperbolic Groups

## Whyburn's Theorem (1958)

*If $D_1, D_2, \ldots$ are disjoint closed topological disks in $S^2$ with $\bigcup_{n\in\mathbb{N}} D_n$ dense and $\mathrm{diam}(D_n) \to 0$, then the complement of their interiors is homeomorphic to the Sierpiński carpet.*



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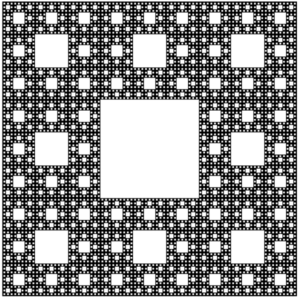

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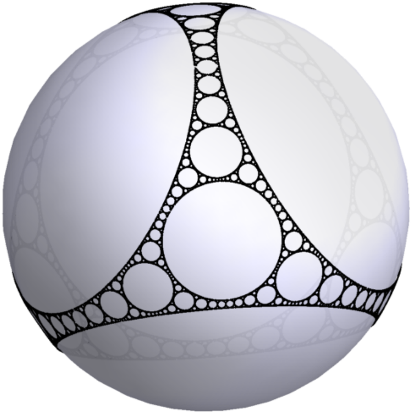
# Quasi-Isometries

## Theorem (Bonk–Schramm 2000)

*Any quasi-isometry $G \longrightarrow H$ between hyperbolic groups induces a quasisymmetry $\partial_\infty G \longrightarrow \partial_\infty H$.*



# Cannon's Conjecture

Let $G$ be a hyperbolic group.

## Cannon's Conjecture (1994)

*If there exists a homeomorphism* $\partial_\infty G \longrightarrow S^2$*, then* $G$ *acts geometrically on* $\mathbb{H}^3$*.*

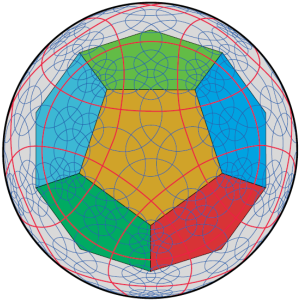


figure from Cannon, Floyd, and Parry 2001

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**Note:** Kapovich–Kleiner (1998) have formulated an analog of Cannon's conjecture for groups with Sierpiński carpet boundary.

# By the Way

## Theorem (Dahmani–Guirardel–Przytycki 2011)

*The boundary of a "random" hyperbolic group is homeomorphic to the Menger sponge.*

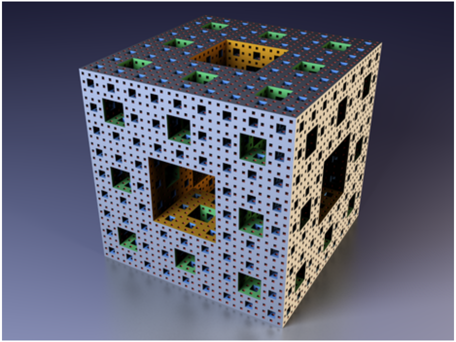

figure by Niabot from Wikimedia Commons

# Quasisymmetries of Sierpiński Carpets

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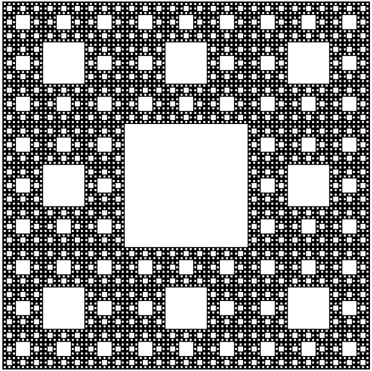
We want to understand quasisymmetries for fractals homeomorphic to the Sierpiński carpet.



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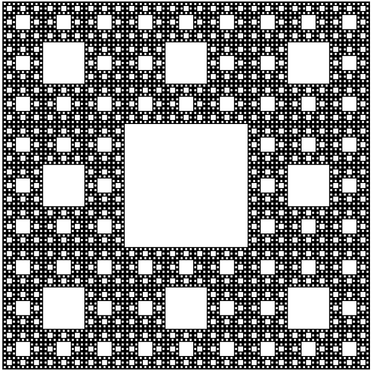
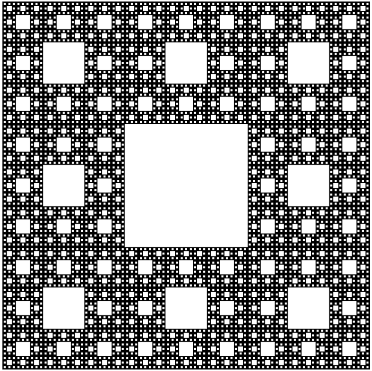


# Quasisymmetries of Sierpiński Carpets

## Theorem (Bonk–Merenkov 2013)

*The quasisymmetry group of the square Sierpiński carpet is dihedral of order 8.*



# Quasisymmetries of Sierpiński Carpets

## Theorem (Bonk–Merenkov 2013)

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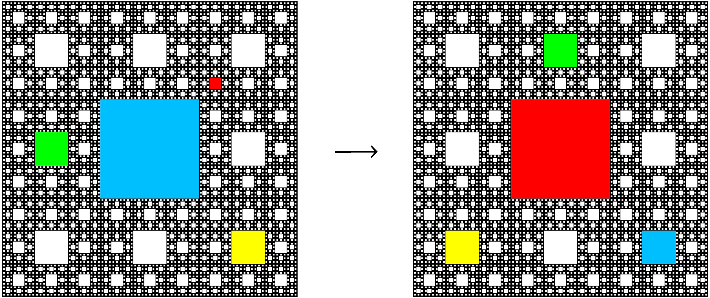
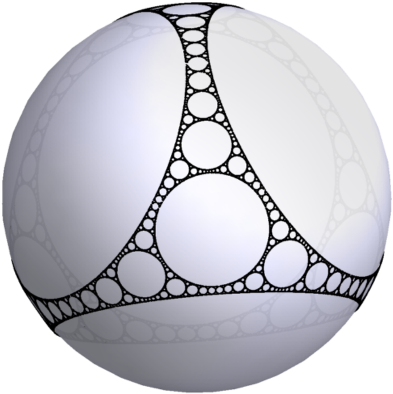


The full homeomorphism group is very large.

# Quasisymmetries of Sierpiński Carpets

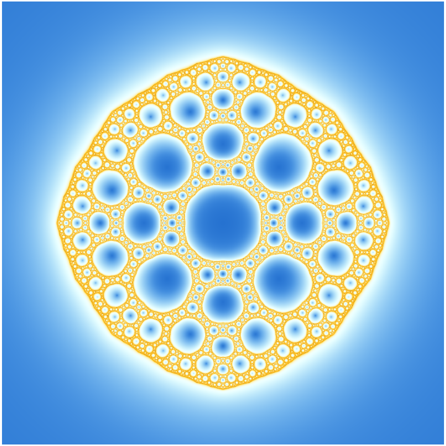
Other Sierpiński carpets can have many quasisymmetries.



So the quasisymmetry group depends on the metric.

# Sierpiński Carpet Julia Sets

Sierpiński carpets also arise as Julia sets for certain rational functions (Milnor–Lei 1993).



$$f(z) = z^2 - \frac{1}{16z^2}$$

# Sierpiński Carpet Julia Sets

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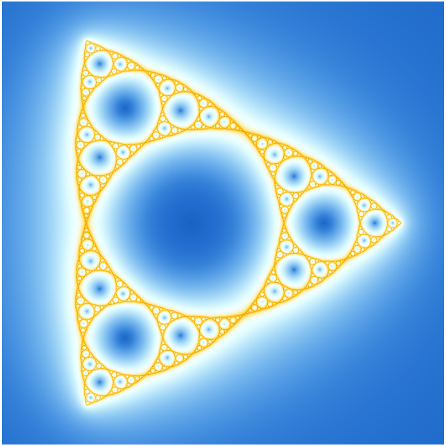
**Theorem (Bonk–Lyubich–Merenkov 2016)**

*Let $f(z)$ be a rational function whose Julia set $J_f$ is a Sierpiński carpet. If $f$ is postcritically finite, then the quasisymmetry group of $J_f$ is finite.*

Qiu, Yang, and Zeng (2019) extend this to a large family of semi-hyperbolic Sierpiński carpet Julia sets.

# A Sierpiński Triangle

Some other Julia sets are also known to have only finitely many quasisymmetries.

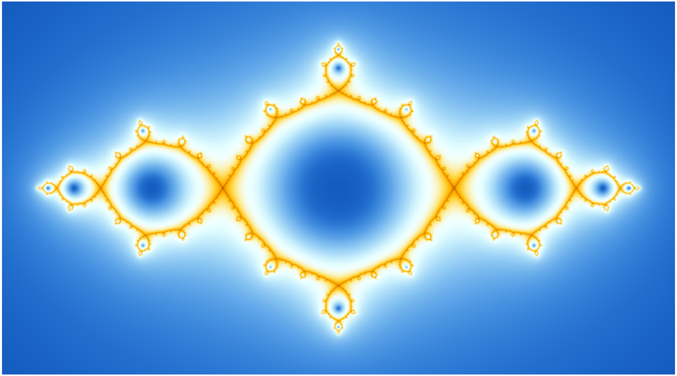


$$f(z) = z^2 - \frac{16}{27z}$$

(Ushiki 1991,
Kameyama 2000)

# The Basilica

The ***basilica*** is the Julia set for $f(z) = z^2 - 1$.



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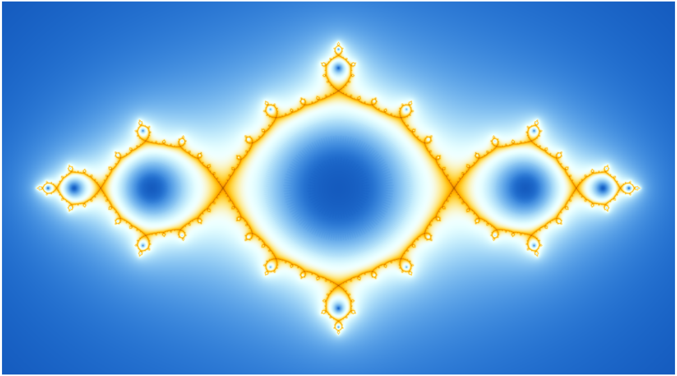
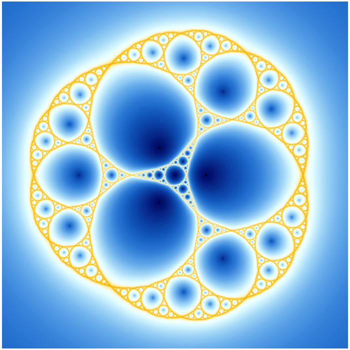


### Theorem (Lyubich–Merenkov 2018)

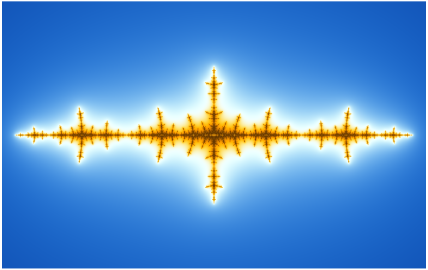
*The quasisymmetry group of the basilica is infinite.*

# Other Examples

Some other Julia sets also have infinitely many quasisymmetries.



Lodge–Lyubich–Merenkov–Mukherjee (2023)



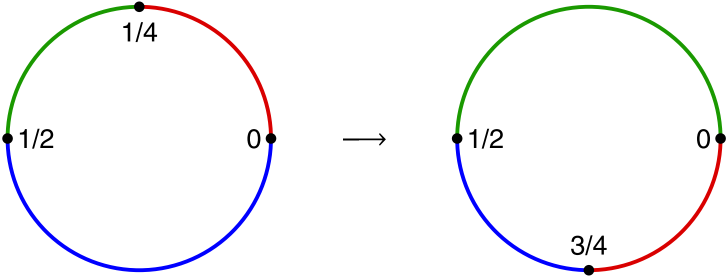
Alland (2023)

# Quasisymmetries of the Basilica

# Quasisymmetries of the Basilica

***Thompson's group*** $T$ is the group of all piecewise-linear homeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$ for which:

1. All slopes are powers of 2, and
2. All breakpoints are dyadic rationals, as is the image of 0.



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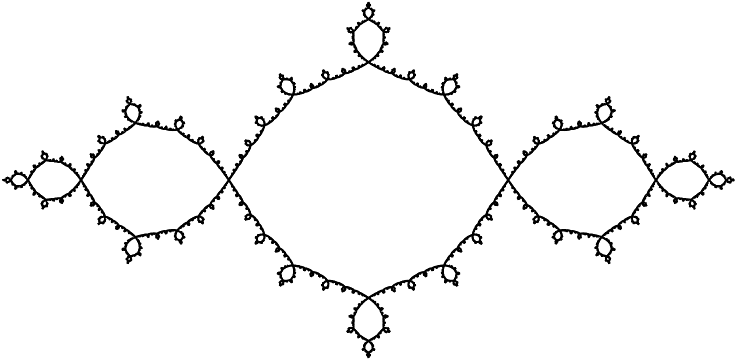
In 2015, Bradley Forrest and I described a natural action of $T$ on the basilica.



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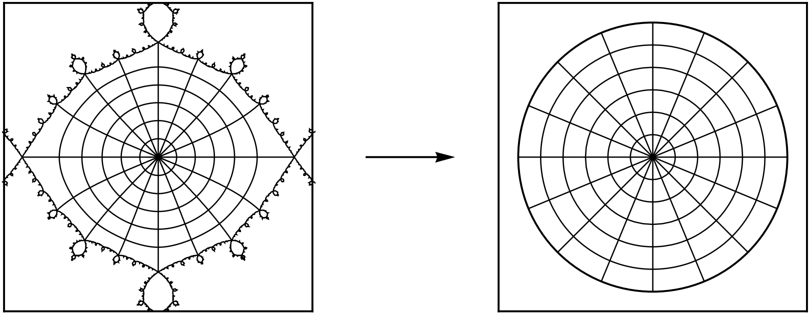

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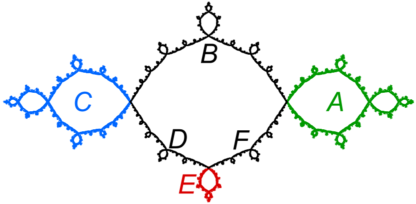
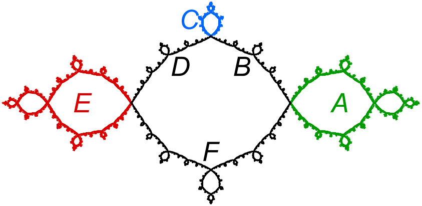
**Domain:**




**Range:**



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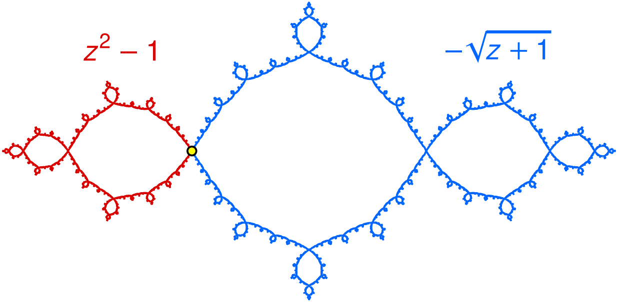
In 2015, Bradley Forrest and I described a natural action of $T$ on the basilica.

This $T$ is contained in a larger group of piecewise-conformal homeomorphisms that we called the ***basilica Thompson group***.



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*The basilica Thompson group is finitely generated, co-embeddable with $T$, and has an index-two subgroup which is simple.*

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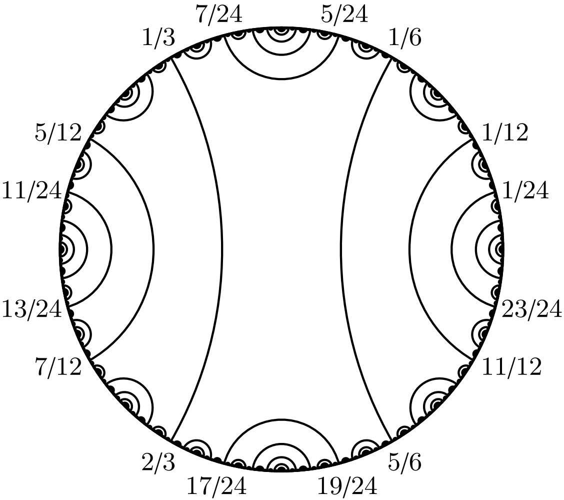
### Theorem (Lyubich–Merenkov 2018)

*All elements of the basilica Thompson group are quasisymmetries.*

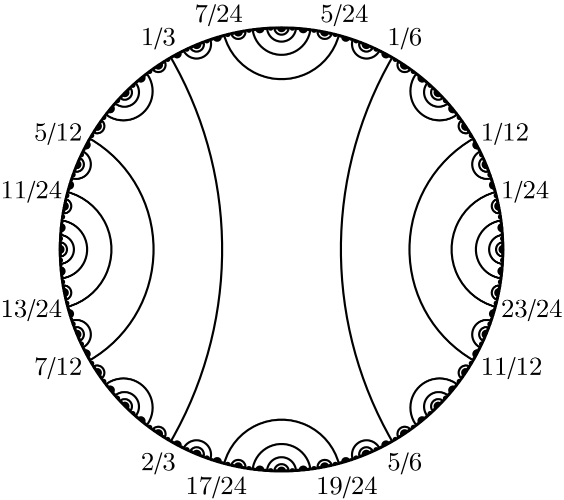
# Quasisymmetries of the Basilica

Their proof uses the Thurston invariant lamination for the basilica.



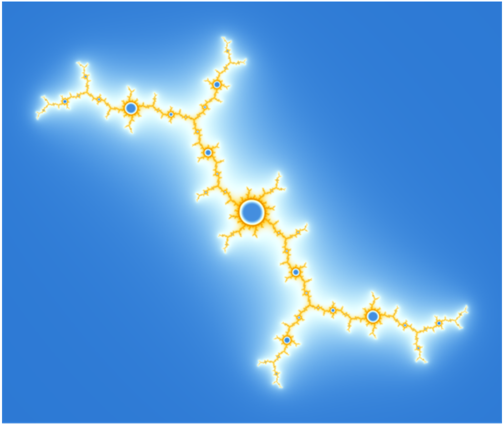
# Quasisymmetries of the Basilica

It depends on the fact that there are only countably many arcs.



# Other Julia Sets

Can we extend this to other Julia sets?



Julia set for $f(z) = z^2 - 0.157 + 1.032i$

# Finitely Ramified Fractals

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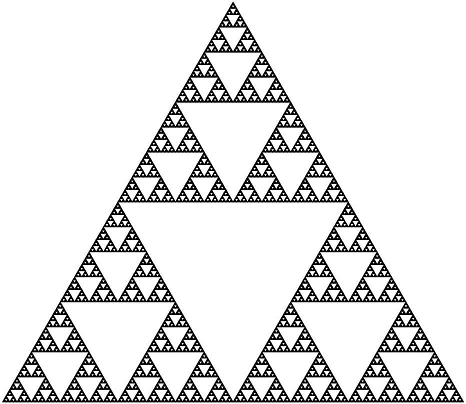
Roughly speaking, a fractal is ***finitely ramified*** if it is made from pieces (called ***cells***) that have finitely many boundary points.



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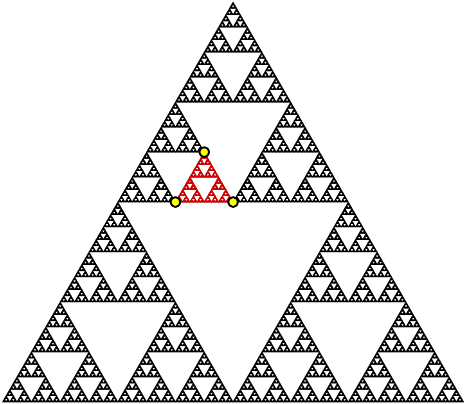

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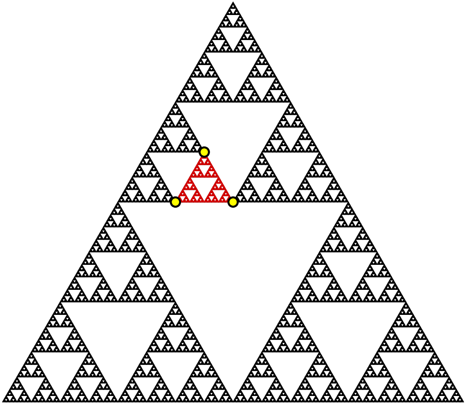

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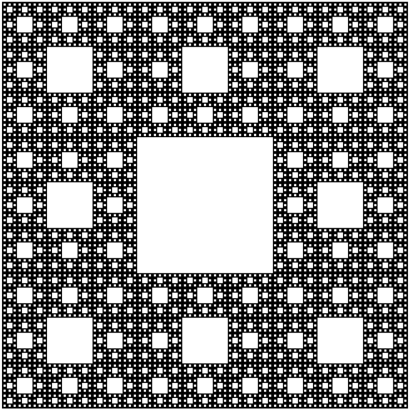

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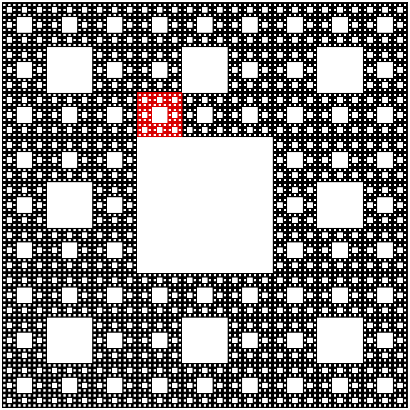

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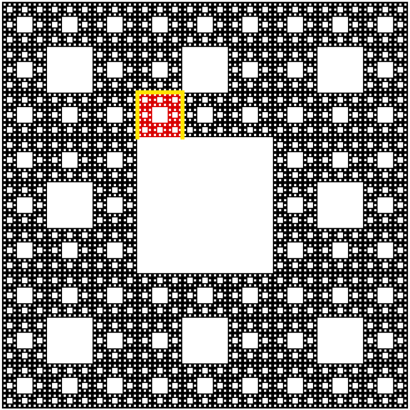

***Not*** finitely ramified

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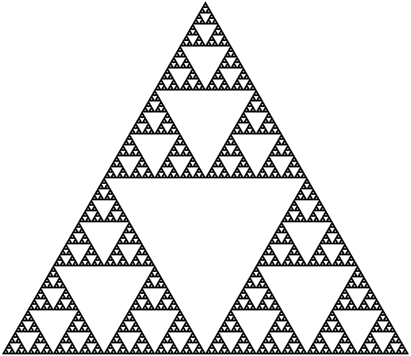

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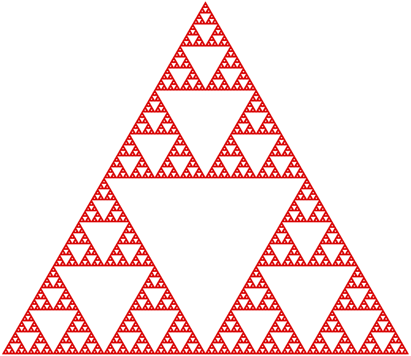

One 0-cell

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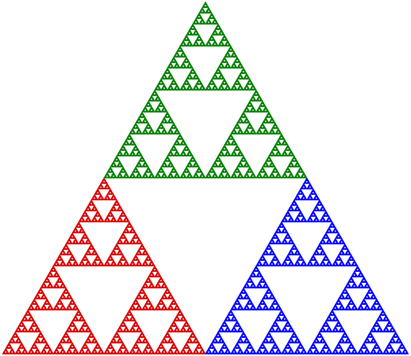

Three 1-cells

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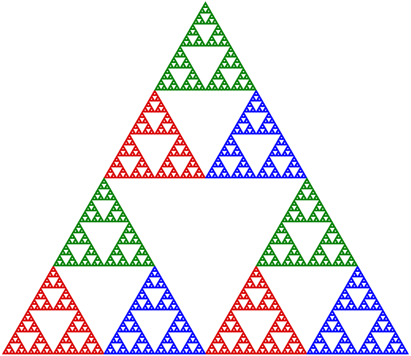

Nine 2-cells

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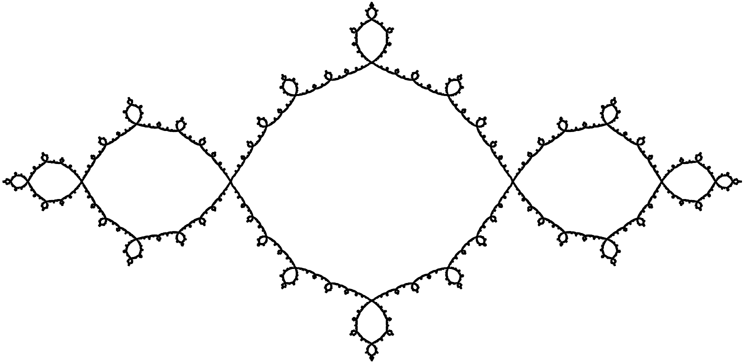
4. If $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ with each $E_n$ an $n$-cell, then $\bigcap_{n=0} E_n$ is a single point.

# Example: The Basilica

The basilica Julia set can be viewed as a finitely ramified fractal.



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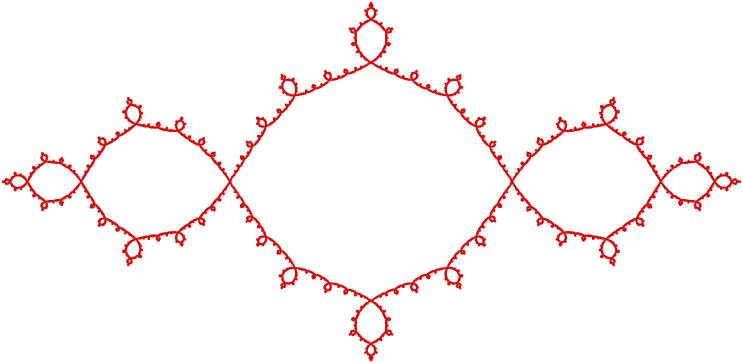

One 0-cell

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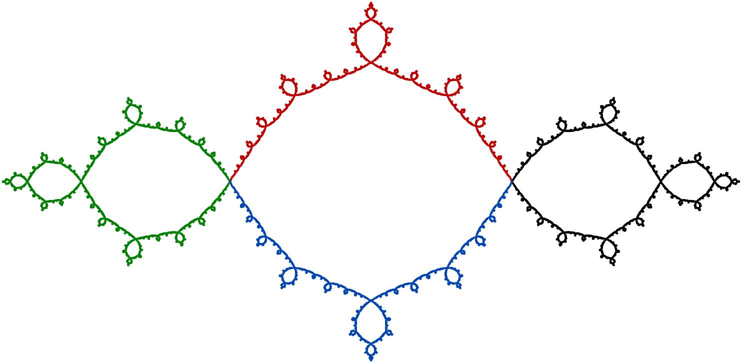

Four 1-cells

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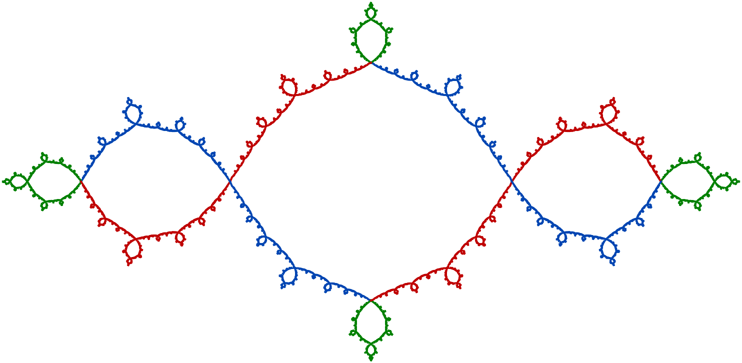

Twelve 2-cells

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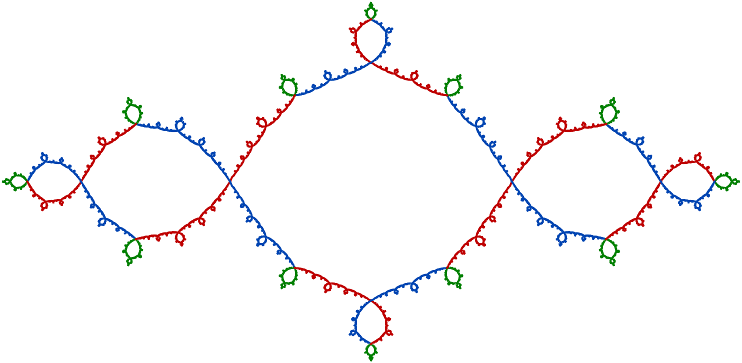
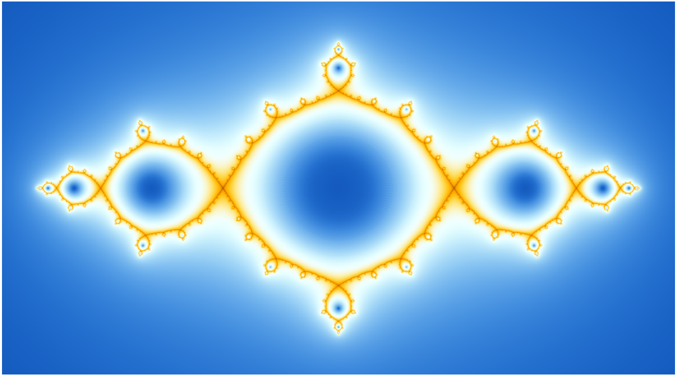


Thirty-six 3-cells

# Finitely Ramified Julia Sets

Julia sets for polynomials tend to be finitely ramified.



Julia set for $f(z) = z^2 - 1$

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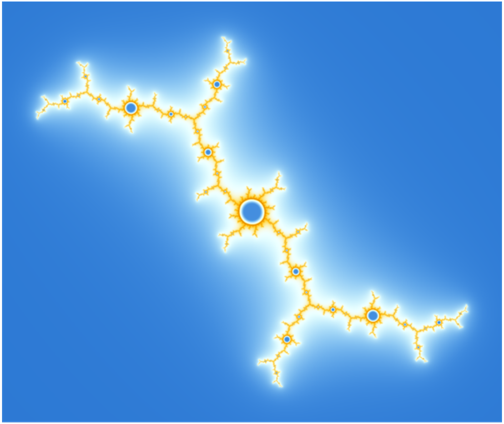

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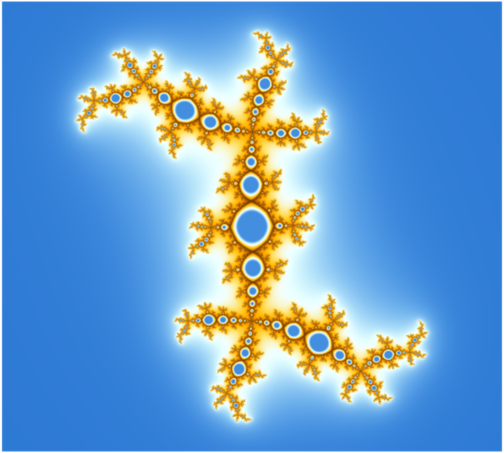

Julia set for $f(z) = z^2 + 0.32 + 0.56i$

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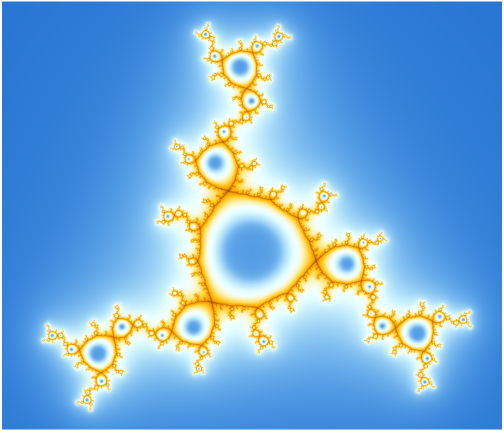
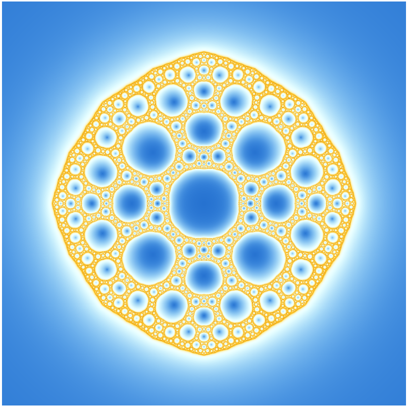


Julia set for $f(z) = z^3 - 0.21 + 1.09i$

# Finitely Ramified Julia Sets

Julia sets for rational functions are sometimes finitely ramified.



Julia set for $f(z) = z^2 - \dfrac{1}{16z^2}$

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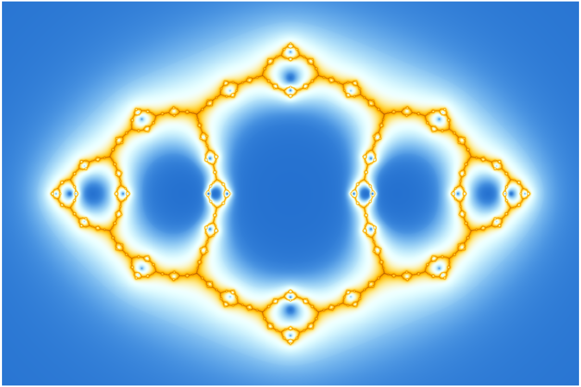

Julia set for $f(z) = \dfrac{1}{z^2} - 1$

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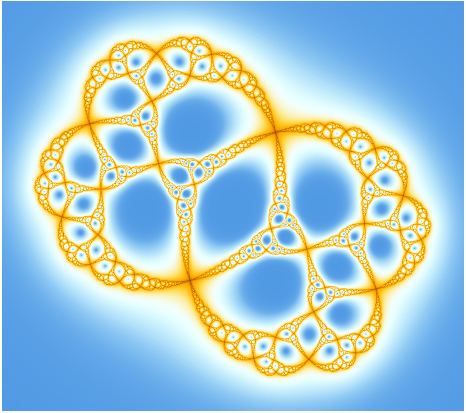

Julia set for $f(z) = \dfrac{e^{2\pi i/3}z^2 - 1}{z^2 - 1}$

# Undistorted Metrics

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A metric on a finitely ramified fractal $X$ is ***undistorted*** if:

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Exponential Decay Condition:

There exist constants $0 < r < R < 1$ and $C \geq 1$ so that

$$\frac{r^{|m-n|}}{C} \leq \frac{\operatorname{diam}(E')}{\operatorname{diam}(E)} \leq CR^{|m-n|}$$

for any $m$-cell $E$ and $n$-cell $E'$ that intersect.

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Cell Separation Condition:

There exists a constant $\delta > 0$ so that

$$d(E, E') \geq \delta \operatorname{diam}(E)$$

for any two $n$-cells $E$ and $E'$ that are disjoint.

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1. *All undistorted metrics on $X$ are quasisymmetrically equivalent.*
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That is, the undistorted metrics on $X$ are a ***quasisymmetric gauge***, determined only by the topology and cell structure.

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**Note:** The theory of undistorted metrics is closely related to the "partitions" studied by Kigami (2018) as well as the "quasi-visual approximations" introduced by Bonk–Meyer (2020) and studied by Bonk–Hlushchanka–Meyer (2026).

# Undistorted Metrics

A metric on a finitely ramified fractal $X$ is ***undistorted*** if:

1. It satisfies the exponential decay condition, and
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### Corollary for Quasisymmetries

*If $X$ and $Y$ have undistorted metrics, a homeomorphism $f : X \longrightarrow Y$ is a quasisymmetry if and only if the pullback of the metric on $Y$ is undistorted.*

# Applications to Julia Sets

# Defining Cells

A set $S \subset J_f$ is a ***branch cut*** if $f^{-1}$ has a single-valued branch on each component of $J_f \setminus S$.

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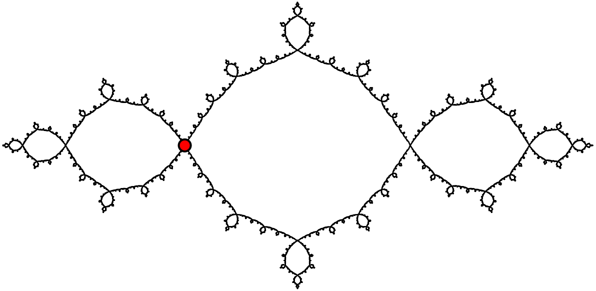

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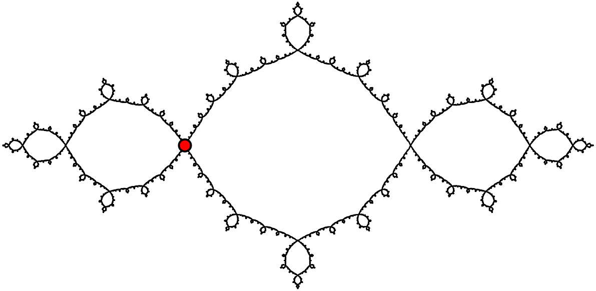

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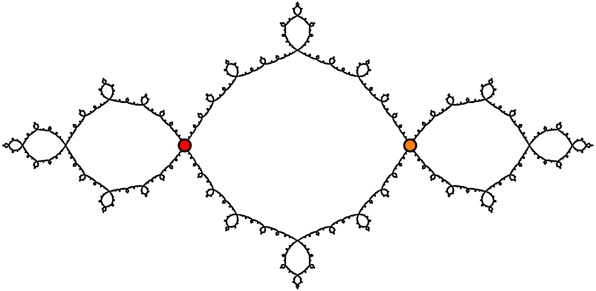

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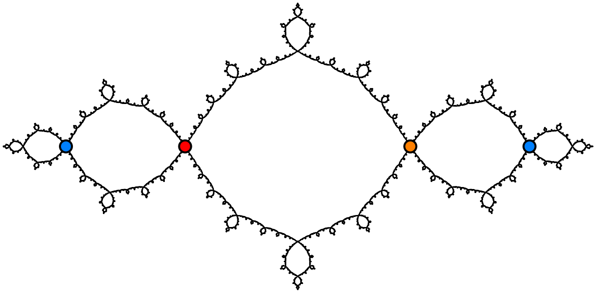

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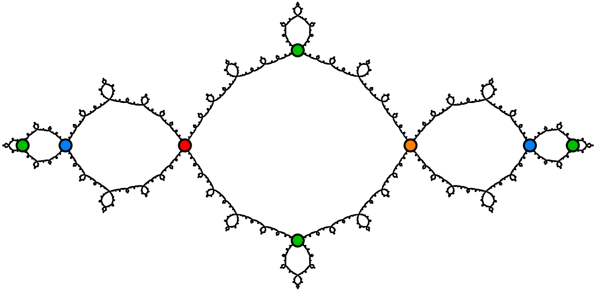

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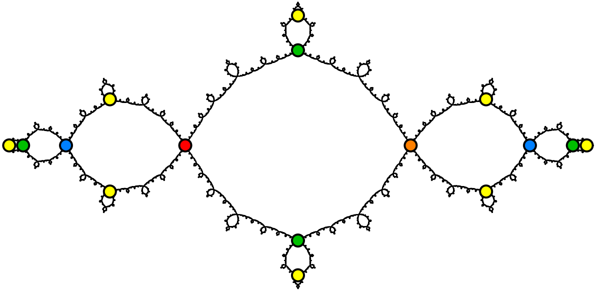

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*If $J_f$ has a finite invariant branch cut, the resulting cells define a finitely ramified cell structure on $J_f$, and the restriction of the spherical metric is undistorted.*

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## Proposition (B–Forrest 2025)

*If $f$ is a hyperbolic polynomial and $J_f$ is connected, then $J_f$ has a finite invariant branch cut.*

# Note: Hyperbolic Only!

We **only** prove the branch cut theorem for $f$ hyperbolic.

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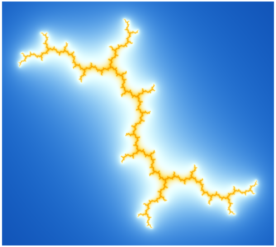
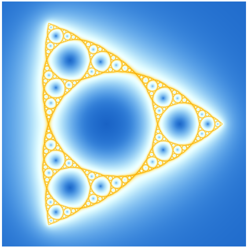
It would be interesting to extend this to sub-hyperbolic $f$.

# Constructing Quasisymmetries

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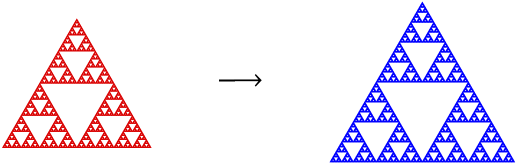
Consider two cells in a finitely ramified fractal $X$:



$E$ ($m$-cell) $\longrightarrow$ $E'$ ($n$-cell)

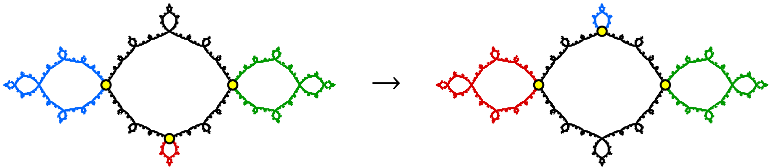
A homeomorphism $E \longrightarrow E'$ is ***cellular*** if it maps $(m+k)$-cells to $(n+k)$-cells for all $k \geq 0$.

# Constructing Quasisymmetries

A homeomorphism of $X$ is ***piecewise-cellular*** if there exist subdivisions

$$\{E_1, \ldots, E_n\} \qquad \text{and} \qquad \{E'_1, \ldots, E'_n\}$$

of $X$ into cells so that each $E_i$ maps to $E'_i$ by a cellular map.



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## Piecewise-Cellular Theorem (B–Forrest 2025)

*If the metric on $X$ is undistorted, then any piecewise-cellular homeomorphism of $X$ is a quasisymmetry.*

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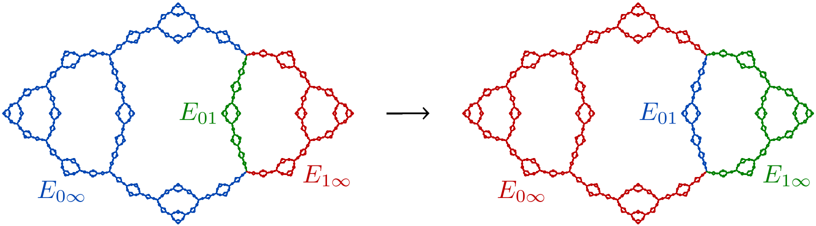
This generalizes Lyubich and Merenkov's result that every element of the basilica Thompson group is a quasisymmetry of the basilica.

# Constructing Quasisymmetries

We can use this theorem to prove that individual Julia sets have lots of quasisymmetries.

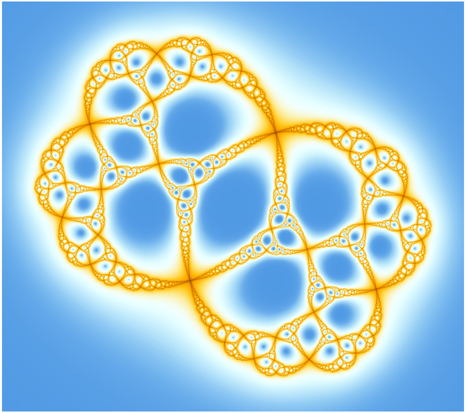



## Theorem (B–Forrest 2025)

*The quasisymmetry group of the Julia set for* $f(z) = \dfrac{1}{z^2} - 1$ *contains* $\mathbb{Z}_3 * \mathbb{Z}_2$.

# Constructing Quasisymmetries

But some finitely ramified Julia sets for hyperbolic rational maps have only finitely many quasisymmetries.



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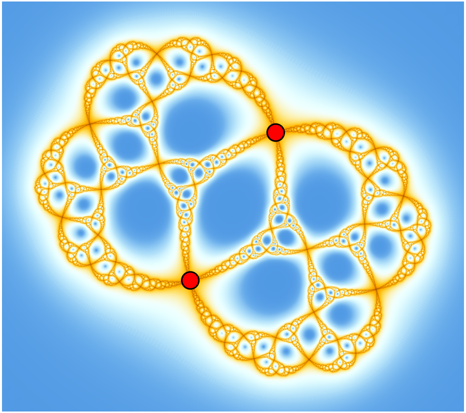

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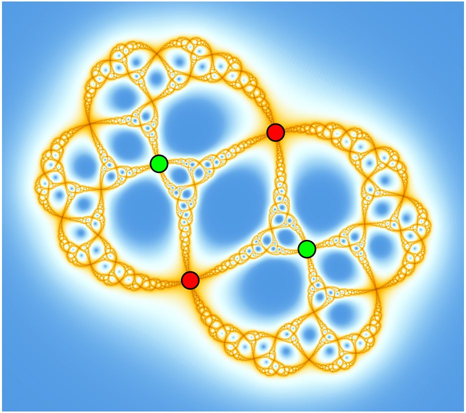

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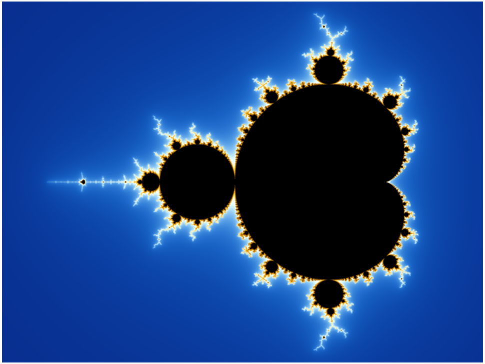
# Polynomials

# Quadratic Polynomials

### Theorem (B–Forrest 2025)

*Any connected Julia set for a hyperbolic quadratic polynomial has infinitely many quasisymmetries.*



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## Theorem (B–Forrest 2025)

*Any connected Julia set for a hyperbolic quadratic polynomial has infinitely many quasisymmetries.*

We have ***two*** proofs of this theorem:

1. We prove that the quasisymmetry group contains $\mathbb{Z}_2 * \mathbb{Z}_n$ for some $n \geq 2$.
2. We also prove that the quasisymmetry group contains Thompson's group $F$.

The two proofs generalize to different classes of higher-degree polynomials.

# Higher-Degree Polynomials

1. If $f$ is a hyperbolic unicritical polynomial of degree $d$ with connected Julia set, then the quasisymmetry group of $J_f$ contains $\mathbb{Z}_d * \mathbb{Z}_n$ for some $n \geq 2$.

2. If $f$ is a postcritically finite polynomial and some leaf of the Hubbard tree lies in a critical cycle of local degree two, then the quasisymmetry group of $J_f$ contains Thompson's group $F$.

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Let's sketch a proof of the first result.

# Making $\mathbb{Z}_2 * \mathbb{Z}_n$

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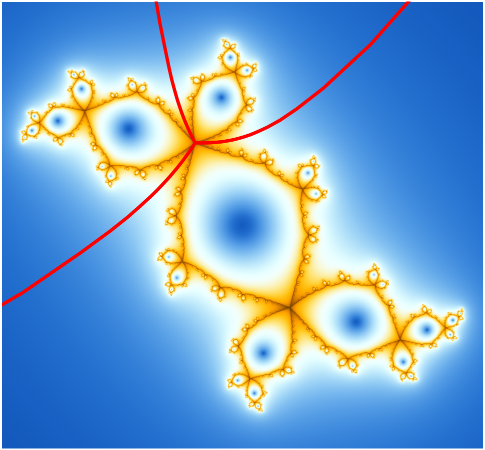
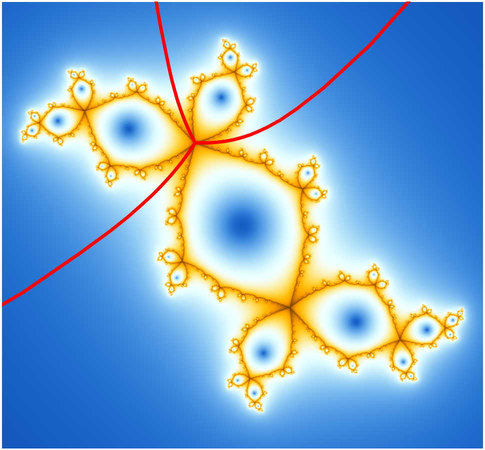
A fixed point for a polynomial $f$ is ***rotational*** if it is the landing point for a nontrivial cycle of external rays.



# Making $\mathbb{Z}_2 * \mathbb{Z}_n$

If the critical points are in the right place, we can define a ***rotation*** at such a point.



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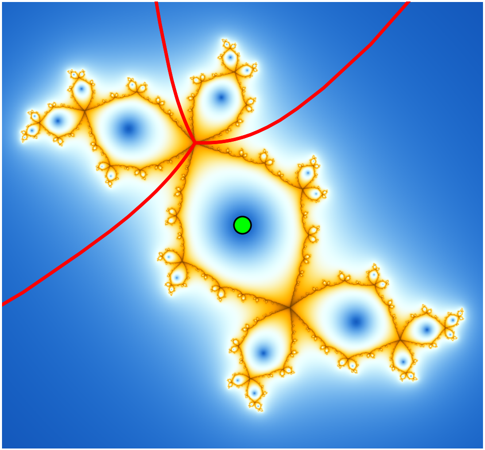

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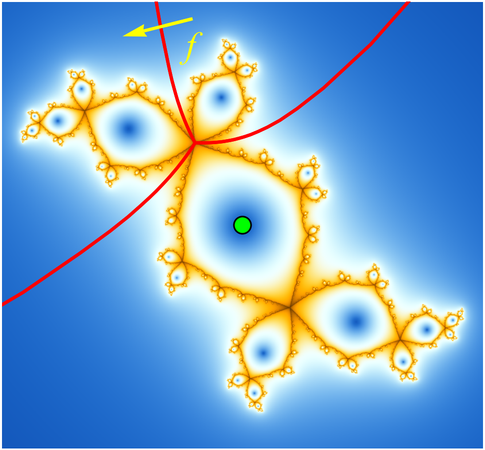

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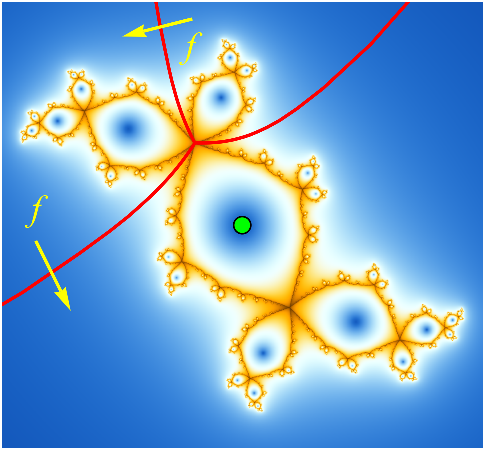

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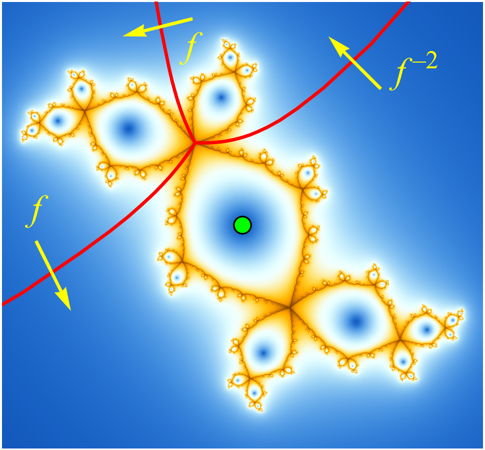
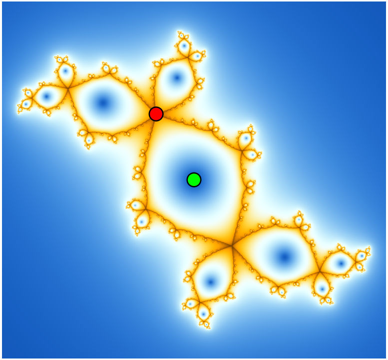
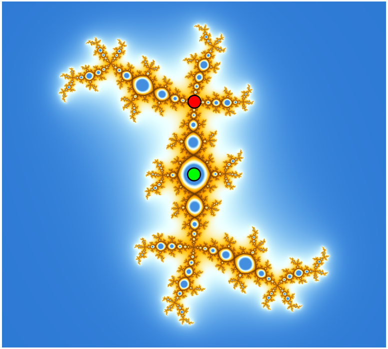


# Making $\mathbb{Z}_d * \mathbb{Z}_n$

Quadratics always have a rotational fixed point, as well as 180° rotational symmetry around the critical point.



contains $\mathbb{Z}_2 * \mathbb{Z}_3$



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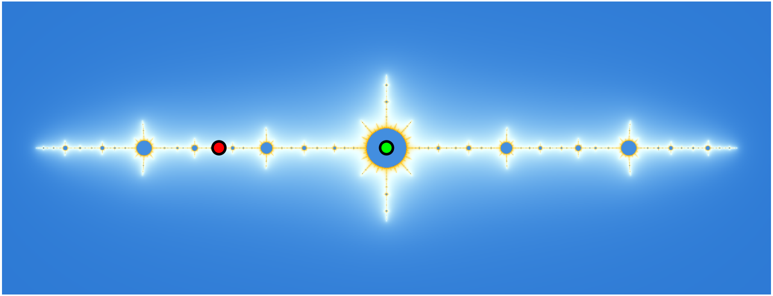

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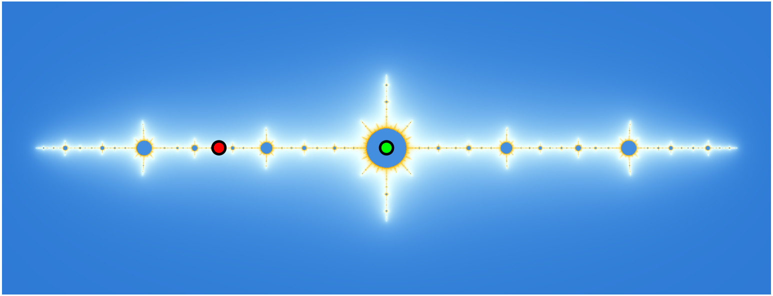
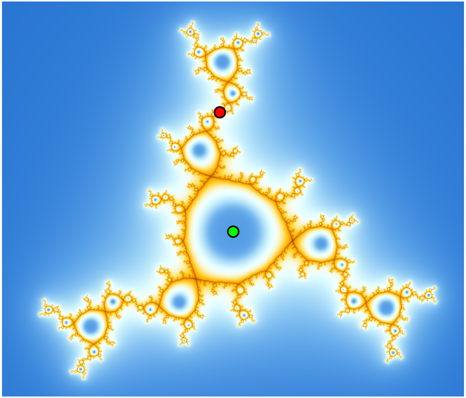


contains $\mathbb{Z}_2 * \mathbb{Z}_2$

The isomorphism with the free product follows from a ping-pong argument.

# Higher Degree

This argument works for unicritical polynomials of any degree. You get $\mathbb{Z}_d * \mathbb{Z}_n$, where $d$ is the degree.



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# Cubic Polynomials

Milnor (1992) classifies hyperbolic cubic polynomials into four types. In the postcritically finite case, these are:

A. Unicritical polynomials.

B. Two critical points in the same cycle.

C. One critical point in a cycle, the other pre-periodic.

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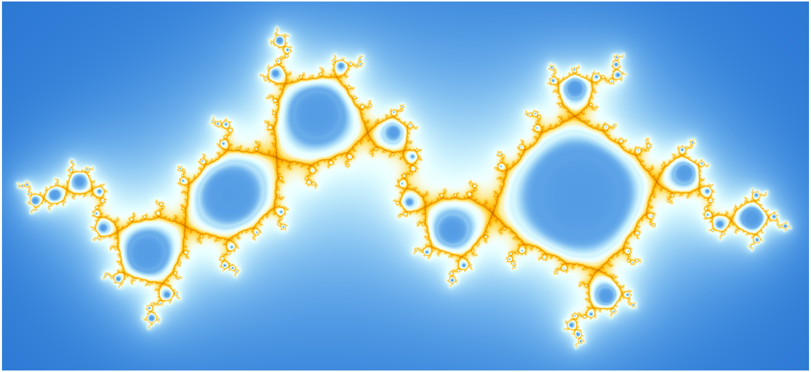
Connected Julia sets of types (A) and (D) always have infinitely many quasisymmetries.

We suspect that there are Julia sets of types (B) and (C) with only finitely many quasisymmetries.

# An Cubic Polynomial of Type (B)

As far as we know, this Julia set only has one nontrivial quasisymmetry.



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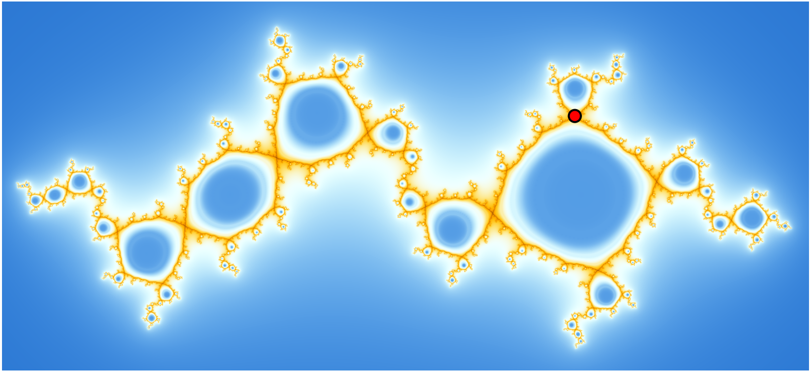

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The End