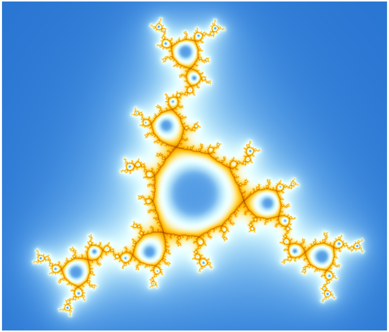
Quasisymmetry Groups of Finitely Ramified Fractals



Jim Belk, University of Glasgow

Séminaire Géométrie Topologie Dynamique
Université Paris-Saclay, 7 May 2026

Joint Work

Bradley Forrest

Stockton University

Quasiconformal Geometry

Quasiconformal Maps

For a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$, let

$$\lfloor T \rfloor = \min_{v \neq 0} \frac{\|Tv\|}{\|v\|} \qquad \text{and} \qquad \lceil T \rceil = \max_{v \neq 0} \frac{\|Tv\|}{\|v\|}$$

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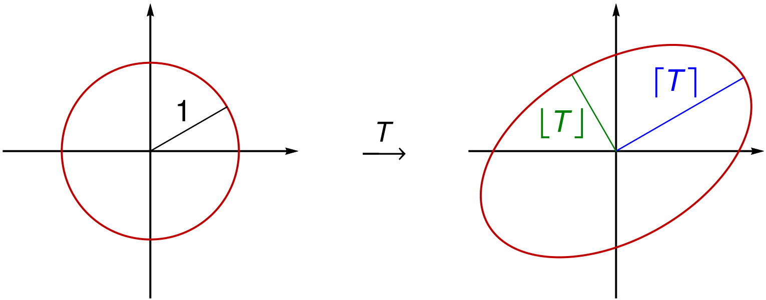

The ratio $\lceil T \rceil / \lfloor T \rfloor$ is a measure of ***eccentricity***.

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A diffeomorphism $f\colon U \longrightarrow U'$ between open subsets of $\mathbb{R}^n$ is ***quasiconformal*** if the function

$$p \longmapsto \frac{\lceil Df_p \rceil}{\lfloor Df_p \rfloor}$$

is bounded on U.

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Note: If $\dfrac{\lceil Df_p \rceil}{\lfloor Df_p \rfloor} \equiv 1$ then f is ***conformal*** (or anticonformal).

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Note 2: This definition can be extended to non-differentiable homeomorphisms.

Applications of Quasiconformal Geometry

- ***Teichmüller theory:*** Defines a metric on the Teichmüller space of a hyperbolic surface (Teichmüller 1940). Leads to a proof of the Nielsen–Thurston classification of mapping classes (Bers 1978).

- ***Mostow rigidity:*** For $n \geq 3$, if X and Y are closed hyperbolic n-manifolds and $\pi_1(X) \cong \pi_1(Y)$ then X and Y are isometric (Mostow 1968).

- ***No wandering domains:*** Every component of the Fatou set for a rational map on the Riemann sphere is periodic or pre-periodic (Sullivan 1985).

Applications of Quasiconformal Geometry

- ***Geometric group theory:*** Any finitely generated group which is quasi-isometric to $\mathbb{H}^n$ has a geometric action on $\mathbb{H}^n$ (Tukia 1986, Gromov 1987, Cannon–Cooper 1992).

- ***Characteristic classes:*** Every n-manifold ($n \neq 4$) supports a unique quasiconformal structure (Sullivan 1978). This allows a theory of characteristic classes for such manifolds (Connes–Sullivan–Teleman 1994).

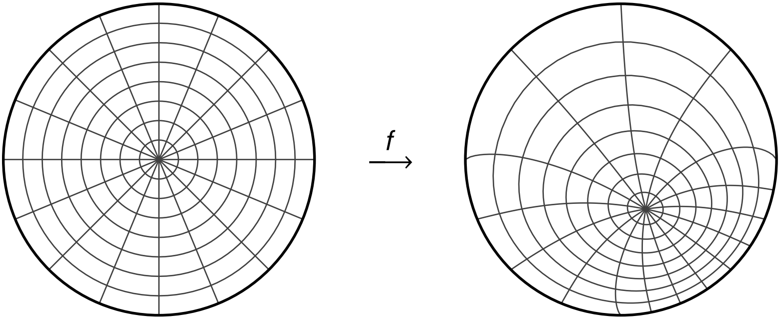
- ***Elliptic PDE's:*** Solution to Calderón's problem on electrical impedance tomography in two dimensions (Astala–Päivärinta 2006).

Quasisymmetries

Quasiconformal Maps on a Disk

Let $f : D^2 \longrightarrow D^2$ be a homeomorphism which is quasiconformal on the interior.

What can the restriction of f to S^1 look like?



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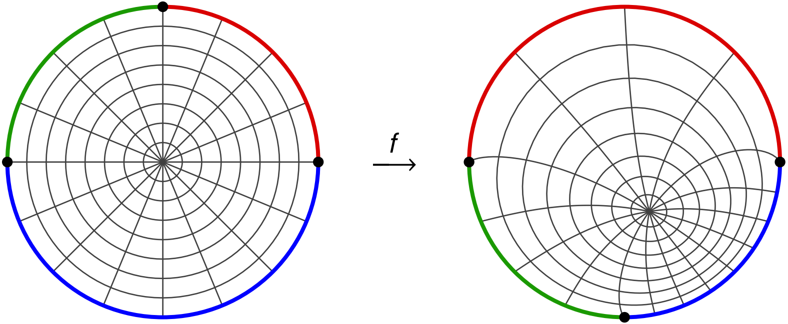

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What can the restriction of f to S^1 look like?

Theorem (Beurling–Ahlfors 1956)

A homeomorphism $f: S^1 \to S^1$ is a restriction of a quasiconformal map on D^2 iff there exists a homeomorphism $\eta: [0,\infty) \to [0,\infty)$ so that

$$\frac{\|f(a) - f(b)\|}{\|f(a) - f(c)\|} \le \eta\left(\frac{\|a - b\|}{\|a - c\|}\right)$$

for every triple a, b, c of distinct points in S^1.

General Definition

Tukia and Väisälä (1980) observed that the Beurling–Ahlfors condition makes sense for homeomorphisms $f : X \longrightarrow Y$ between arbitrary metric spaces.

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Definition

A homeomorphism $f\colon X \to Y$ is a ***quasisymmetry*** if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

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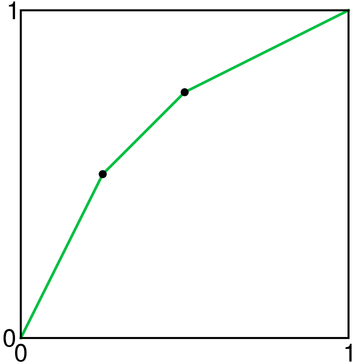
Note: The quasisymmetries $X \longrightarrow X$ form a group.

Examples



$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

If f is bilipschitz with

$$\frac{1}{K}\, d(x,x') \le d\big(f(x),f(x')\big) \le K\, d(x,x')$$

then f is quasisymmetric with $\eta(t) = K^2 t$.

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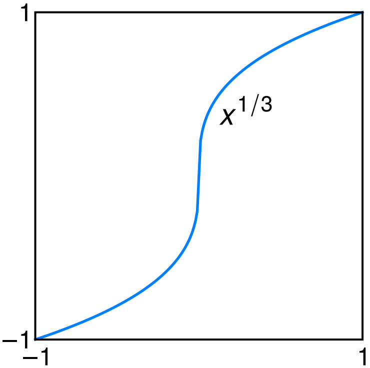

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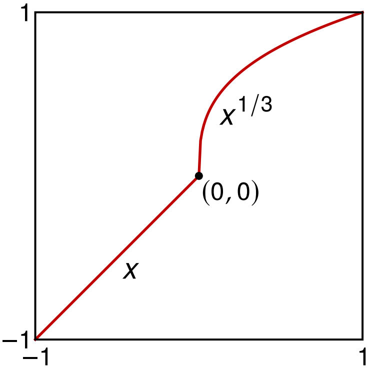
The function $f(x) = x^{1/3}$ is a quasisymmetry of $[-1, 1]$, with

$$\eta(t) = \begin{cases} 6t^{1/3} & \text{if } 0 \leq t \leq 1 \\ 6t & \text{if } t > 1. \end{cases}$$

A Non-Example



$$\frac{d\big(f(a),f(b)\big)}{d\big(f(a),f(c)\big)} \le \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

This function is **not** a quasisymmetry of $[-1,1]$.

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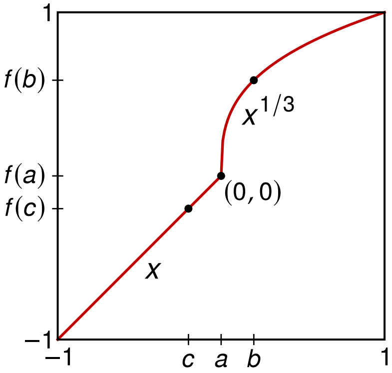


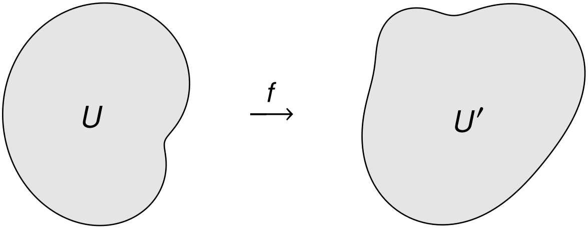
$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} \leq \eta\left(\frac{d(a,b)}{d(a,c)}\right)$$

For $a = 0$, $b = \varepsilon$, and $c = -\varepsilon$, we have

$$\frac{d\big(f(a), f(b)\big)}{d\big(f(a), f(c)\big)} = \frac{\varepsilon^{1/3}}{\varepsilon} = \frac{1}{\varepsilon^{2/3}} \qquad \text{and} \qquad \frac{d(a,b)}{d(a,c)} = 1.$$

Quasiconformal vs. Quasisymmetric

Let $f : U \to U'$ be a homeomorphism between domains in $\mathbb{R}^n$.



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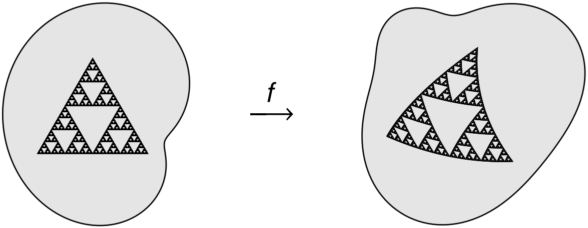

Theorem (Väisälä 1981)

If f is quasiconformal then f restricts to a quasisymmetry on every compact subset of U.

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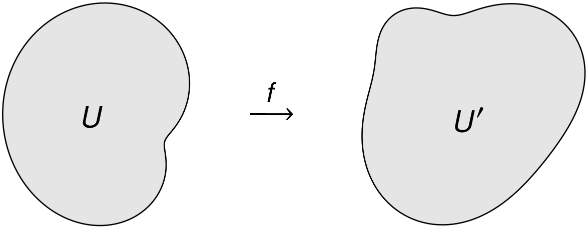

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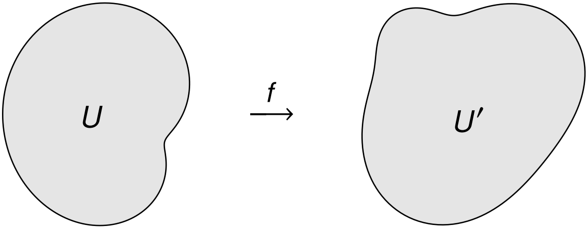

Theorem (Egg Yolk Principle, Väisälä 1981)

The following are equivalent:

1. *f is quasiconformal.*
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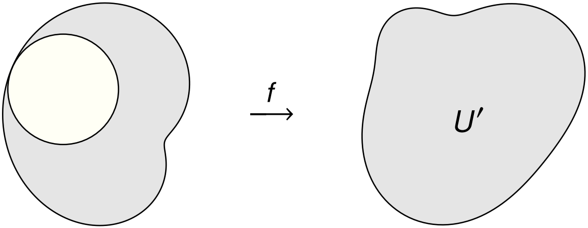

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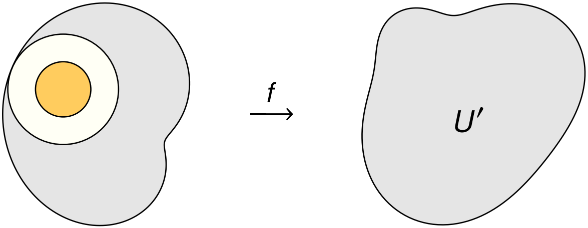

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Relation to Hyperbolic Groups

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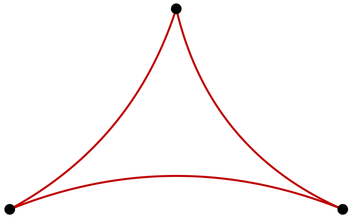

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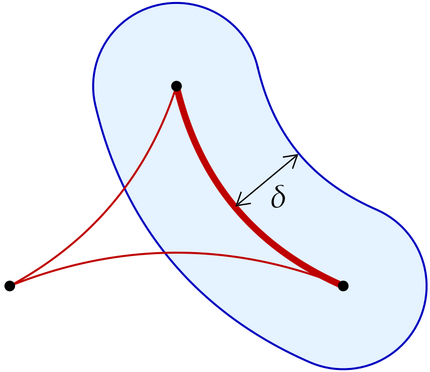

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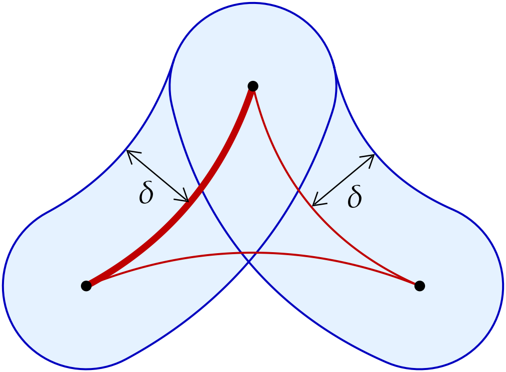

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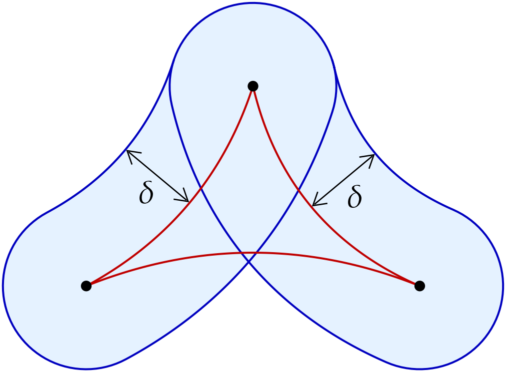

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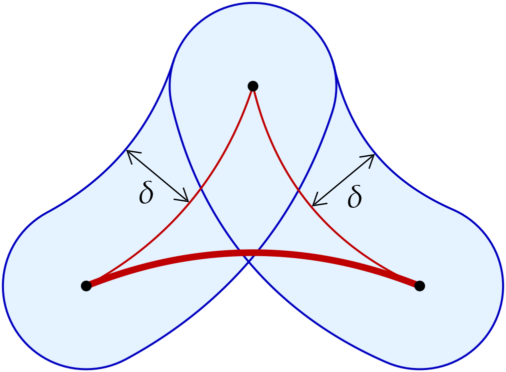

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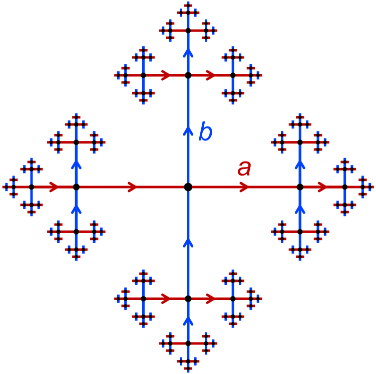

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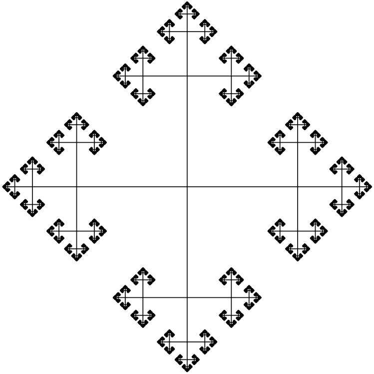

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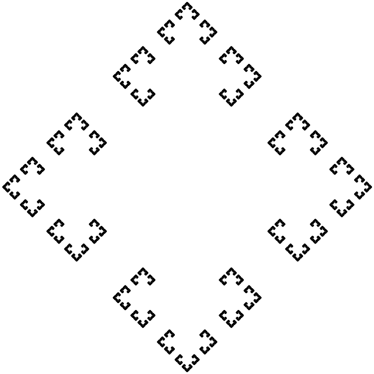

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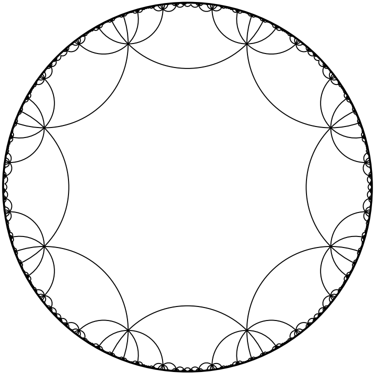

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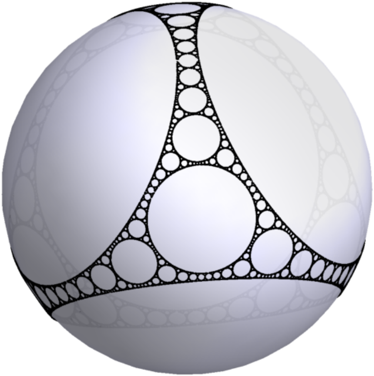

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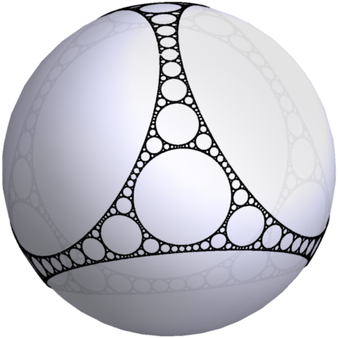

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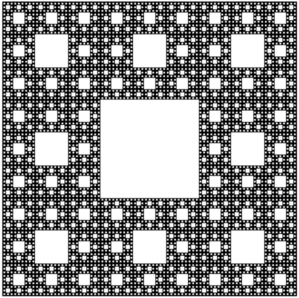
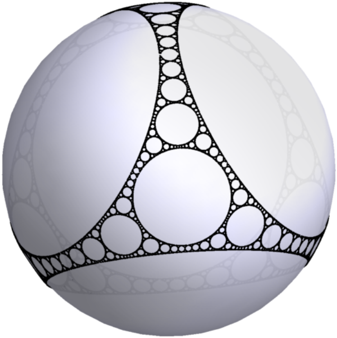


Sierpiński carpet

Hyperbolic Groups

Whyburn's Theorem (1958)

If $D_1, D_2, \ldots$ are disjoint closed topological disks in S^2 with $\bigcup_{n\in\mathbb{N}} D_n$ dense and $\mathrm{diam}(D_n) \to 0$, then the complement of their interiors is homeomorphic to the Sierpiński carpet.



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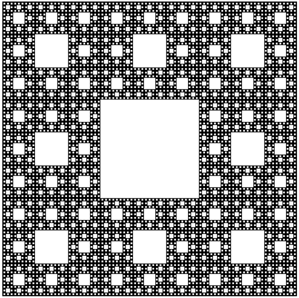
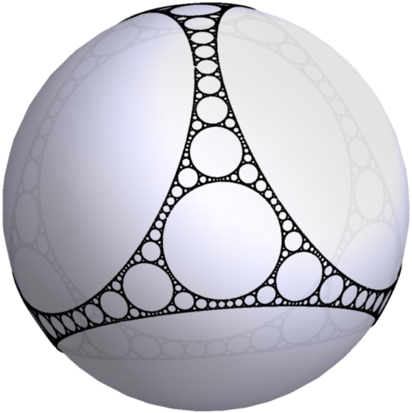


Sierpiński carpet

Quasi-Isometries

Theorem (Bonk–Schramm 2000)

Any quasi-isometry $G \longrightarrow H$ between hyperbolic groups induces a quasisymmetry $\partial_\infty G \longrightarrow \partial_\infty H$.



Cannon's Conjecture

Let G be a hyperbolic group.

Cannon's Conjecture (1994)

If there exists a homeomorphism $\partial_\infty G \longrightarrow S^2$, then G acts geometrically on $\mathbb{H}^3$.

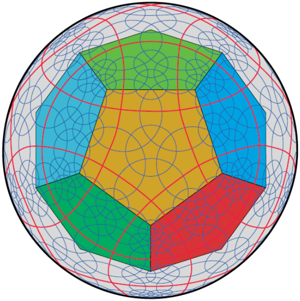


figure from Cannon, Floyd, and Parry 2001

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Theorem (Sullivan–Tukia 1986)

If there exists a ***quasisymmetry*** *$\partial_\infty G \to S^2$ then G acts geometrically on $\mathbb{H}^3$.*

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Theorem (Sullivan–Tukia 1986)

*If there exists a **quasisymmetry** $\partial_\infty G \to S^2$ then G acts geometrically on $\mathbb{H}^3$.*

Note: Kapovich–Kleiner (1998) have formulated an analog of Cannon's conjecture for groups with Sierpiński carpet boundary.

By the Way

Theorem (Dahmani–Guirardel–Przytycki 2011)

The boundary of a "random" hyperbolic group is homeomorphic to the Menger sponge.

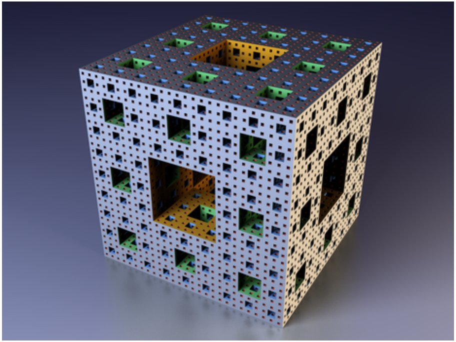

figure by Niabot from Wikimedia Commons

Quasisymmetries of Sierpiński Carpets

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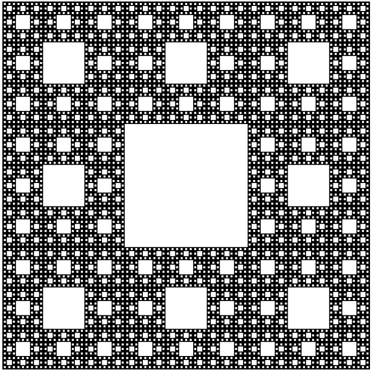
We want to understand quasisymmetries for fractals homeomorphic to the Sierpiński carpet.



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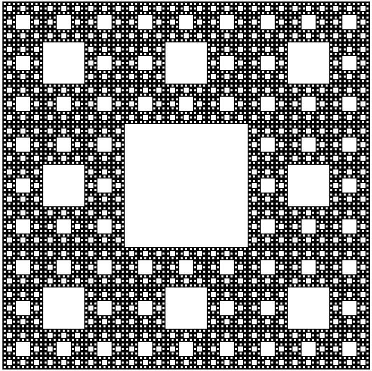

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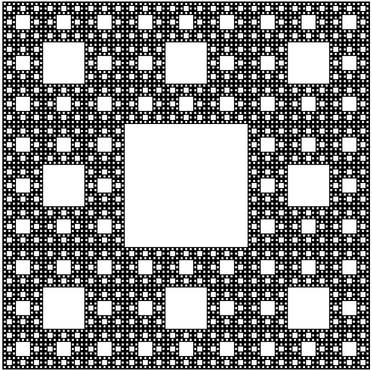
Theorem (Bonk–Merenkov 2013)

The quasisymmetry group of the square Sierpiński carpet is dihedral of order 8.



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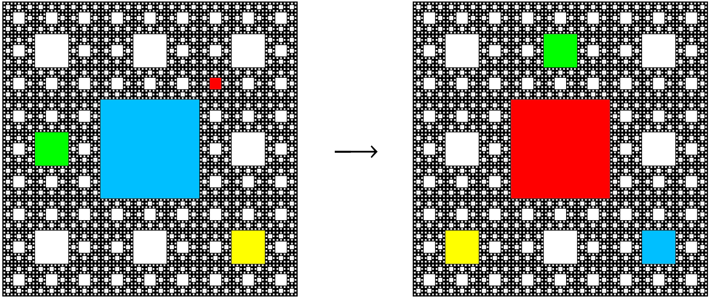
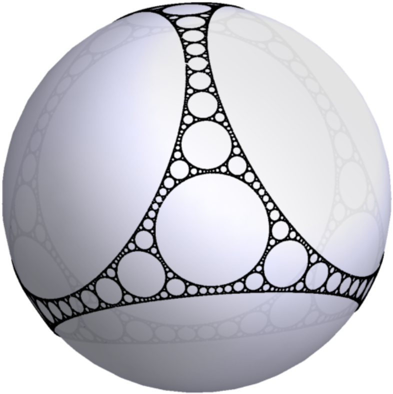


The full homeomorphism group is very large.

Quasisymmetries of Sierpiński Carpets

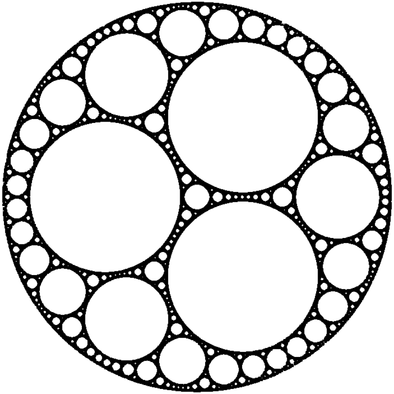
Other Sierpiński carpets can have many quasisymmetries.



So the quasisymmetry group depends on the metric.

Quasisymmetries of Sierpiński Carpets

A ***round carpet*** is a Sierpiński carpet whose holes are round disks.



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Rigidity Theorem (Bonk–Kleiner–Merenkov 2009)

Any quasisymmetry between round carpets of Lebesgue measure zero must be a Möbius transformation.

In particular, the quasisymmetry group of such a carpet is the group of conformal homeomorphisms.

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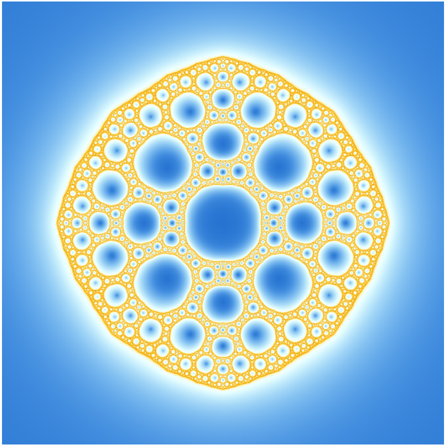
Uniformization Theorem (Bonk 2011)

A Sierpiński carpet is quasisymmetrically equivalent to a round carpet if and only if:

1. *The holes are uniform quasicircles, and*
2. *The holes are uniformly relatively separated.*

Sierpiński Carpet Julia Sets

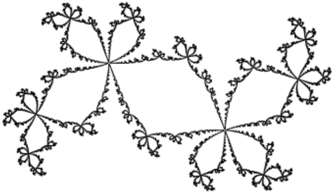
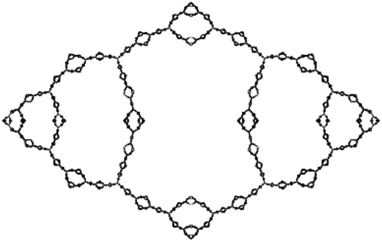
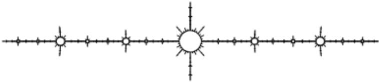
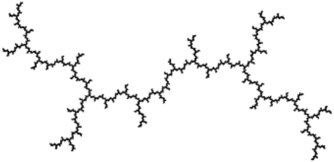
Sierpiński carpets also arise as Julia sets for certain rational functions (Milnor–Lei 1993).



$$f(z) = z^2 - \frac{1}{16z^2}$$

Julia Sets

Every rational function on the Riemann sphere has a ***Julia set*** (the closure of the repelling periodic points).

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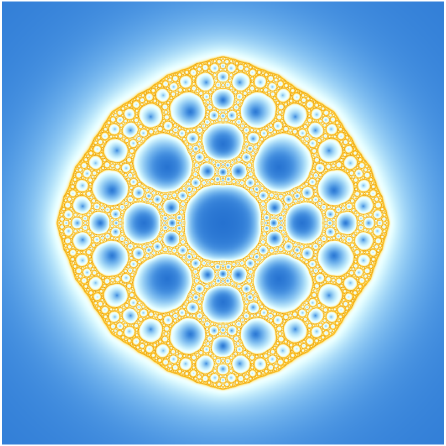

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Sierpiński Carpet Julia Sets

Sierpiński carpets also arise as Julia sets for certain rational functions (Milnor–Lei 1993).

Theorem (Bonk–Lyubich–Merenkov 2016)

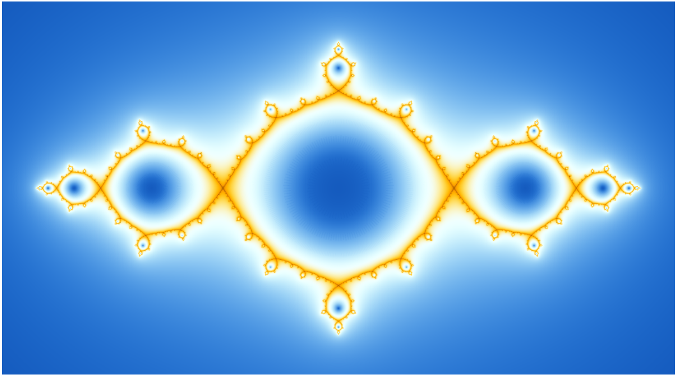
Let $f(z)$ be a rational function whose Julia set J_f is a Sierpiński carpet. If f is postcritically finite, then the quasisymmetry group of J_f is finite.

Qiu, Yang, and Zeng (2019) extend this to a large family of semi-hyperbolic Sierpiński carpet Julia sets.

Quasisymmetries of the Basilica

The Basilica

The ***basilica*** is the Julia set for $f(z) = z^2 - 1$



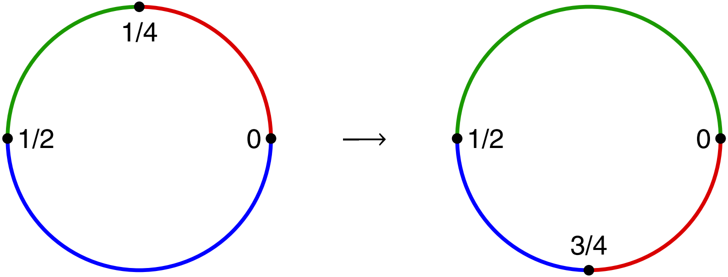
Theorem (Lyubich–Merenkov 2018)

The quasisymmetry group of the basilica is infinite.

Quasisymmetries of the Basilica

Thompson's group T is the group of all piecewise-linear homeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$ for which:

1. All slopes are powers of 2, and
2. All breakpoints are dyadic rationals, as is the image of 0.



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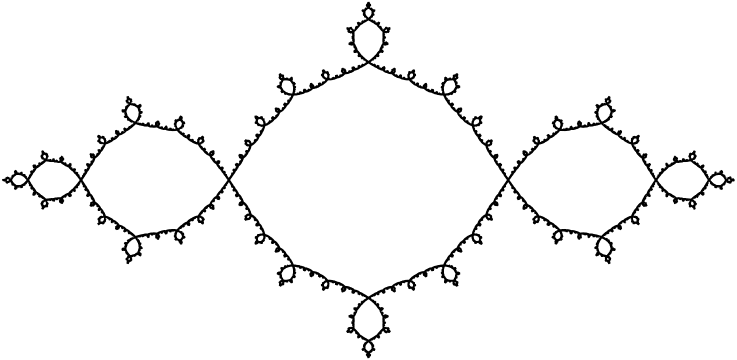

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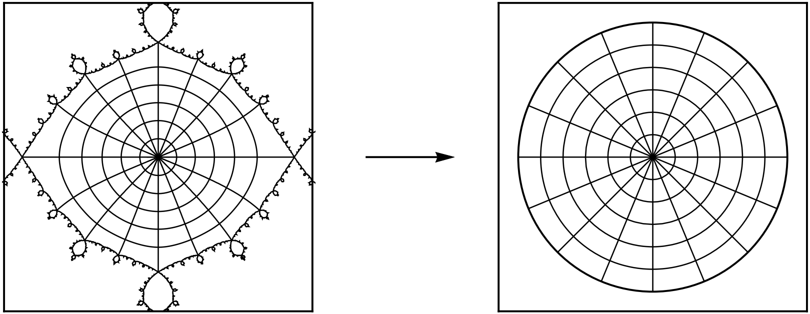

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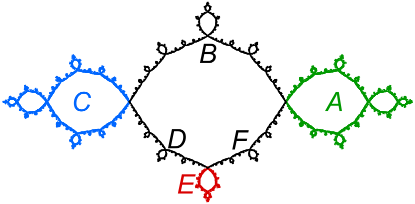
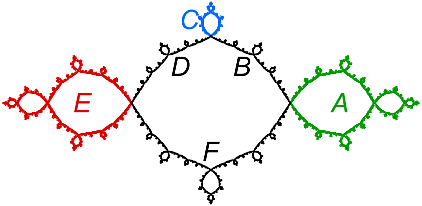
Domain:



Range:



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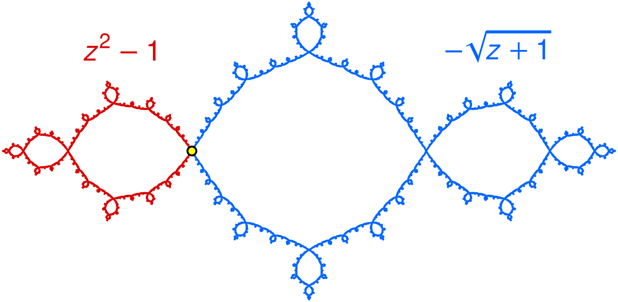
This T is contained in a larger group of piecewise-conformal homeomorphisms that we called the ***basilica Thompson group***.



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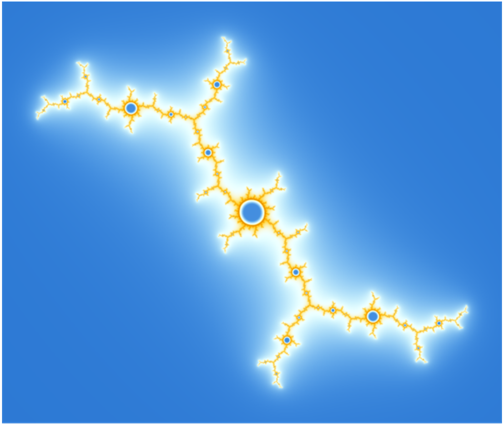
Theorem (Lyubich–Merenkov 2018)

All elements of the basilica Thompson group are quasisymmetries.

Other Julia Sets

Can we extend this to other Julia sets?

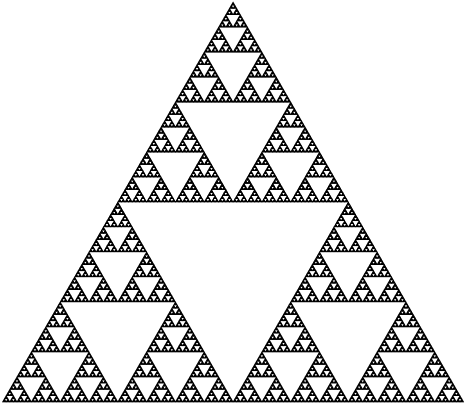


Julia set for $f(z) = z^2 - 0.157 + 1.032i$

Finitely Ramified Fractals

Finitiely Ramified Fractals

Roughly speaking, a fractal is ***finitely ramified*** if it is made from pieces (called ***cells***) that have finitely many boundary points.



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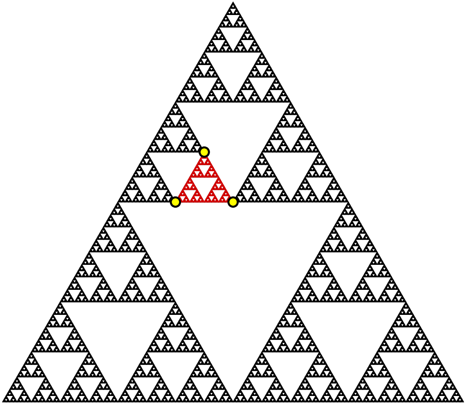

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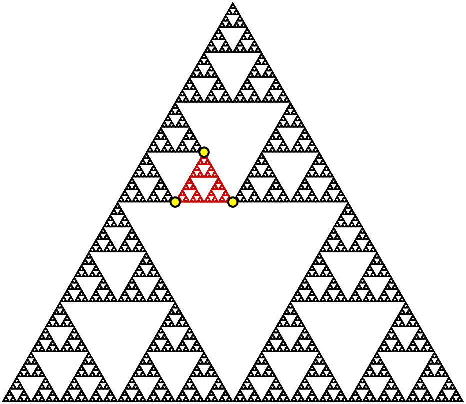

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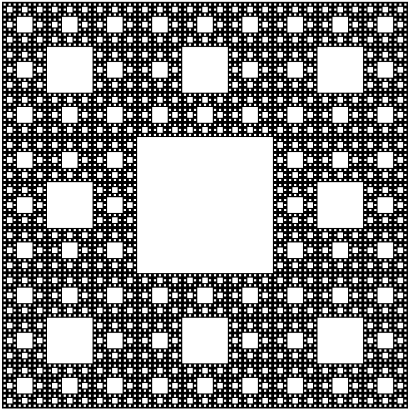

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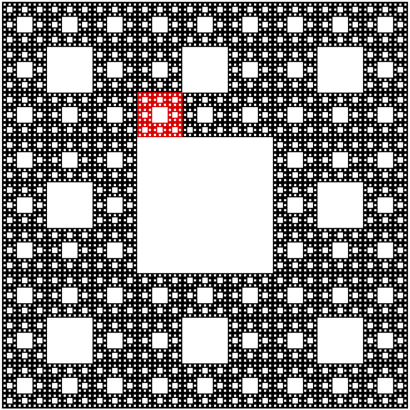

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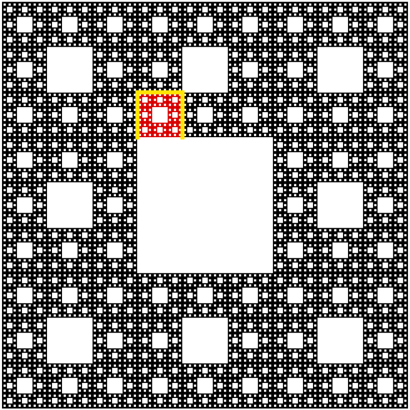

Not finitely ramified

General Definition

Definition (Teplyaev 2008)

Let X be a compact, connected metrizable space.

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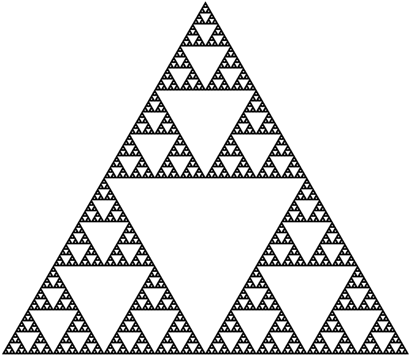

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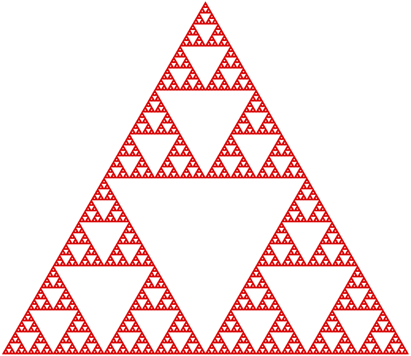

One 0-cell

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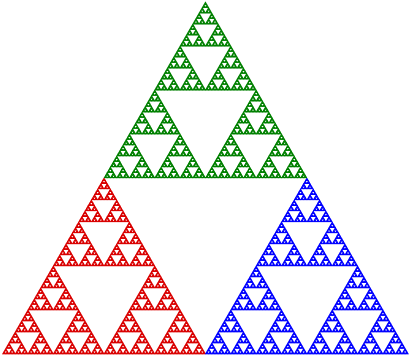

Three 1-cells

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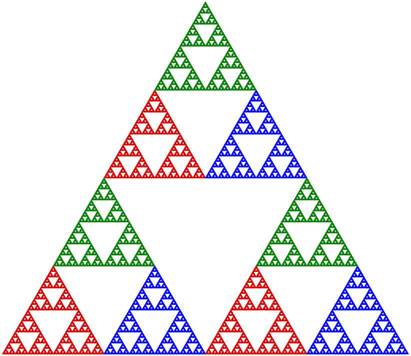

Nine 2-cells

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2.
3.
4.

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2. The intersection of any two n-cells is finite.
3. The entire space X is the unique 0-cell, and every n-cell is a union of $(n+1)$-cells.
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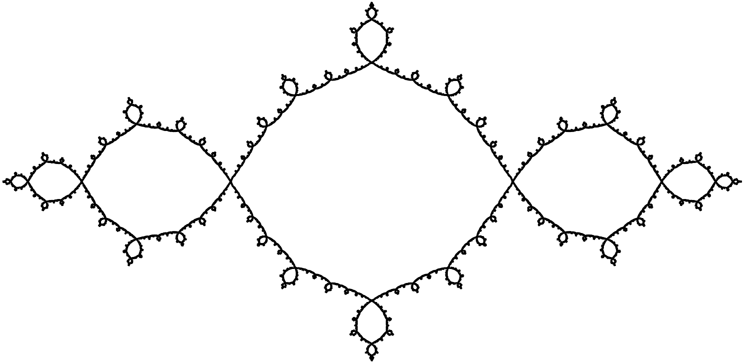
2. The intersection of any two n-cells is finite.
3. The entire space X is the unique 0-cell, and every n-cell is a union of $(n+1)$-cells.
4. If $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ with each E_n an n-cell, then $\bigcap_{n=0} E_n$ is a single point.

Example: The Basilica

The basilica Julia set can be viewed as a finitely ramified fractal.



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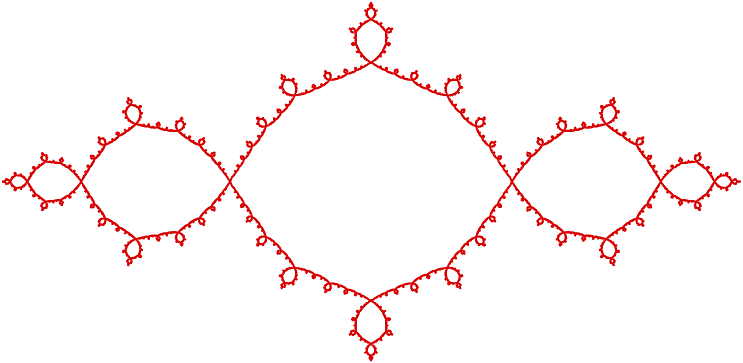

One 0-cell

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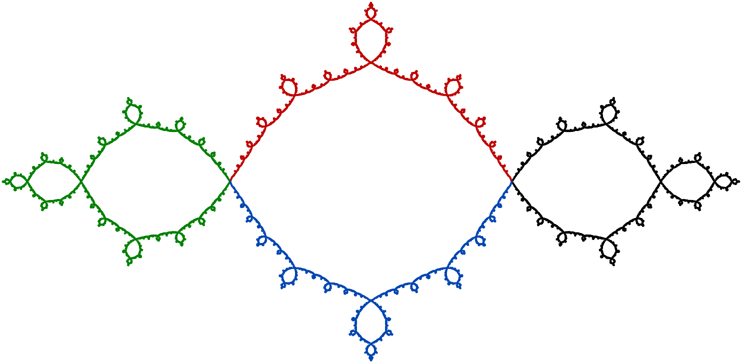

Four 1-cells

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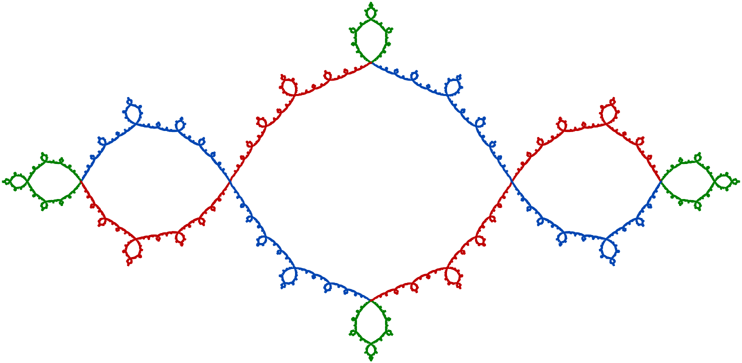

Twelve 2-cells

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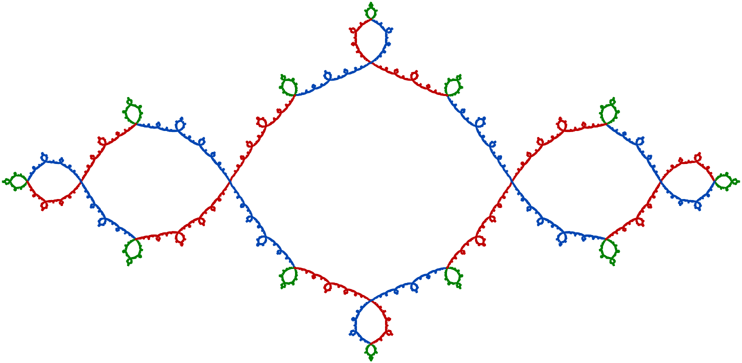
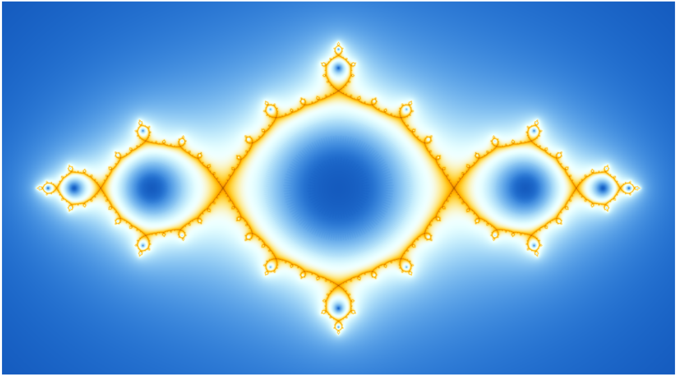


Thirty-six 3-cells

Finitely Ramified Julia Sets

Julia sets for polynomials tend to be finitely ramified.



Julia set for $f(z) = z^2 - 1$

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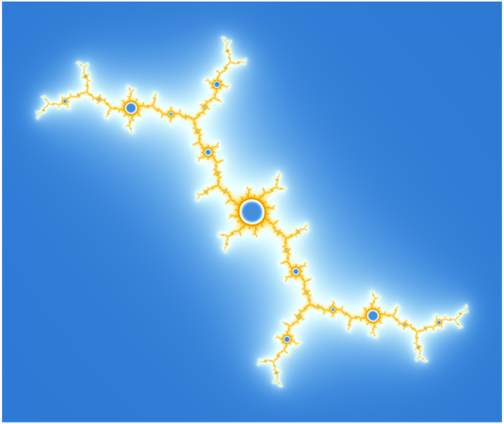

Julia set for $f(z) = z^2 - 0.157 + 1.032i$

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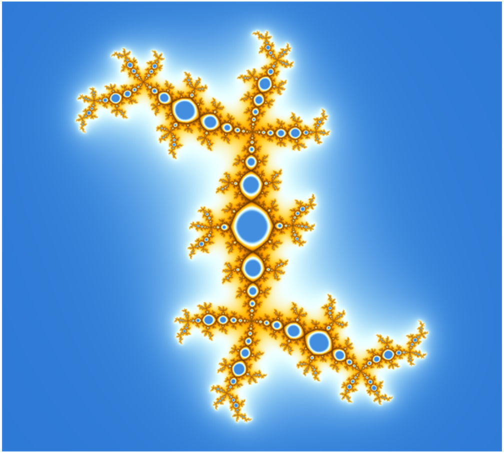

Julia set for $f(z) = z^2 + 0.32 + 0.56i$

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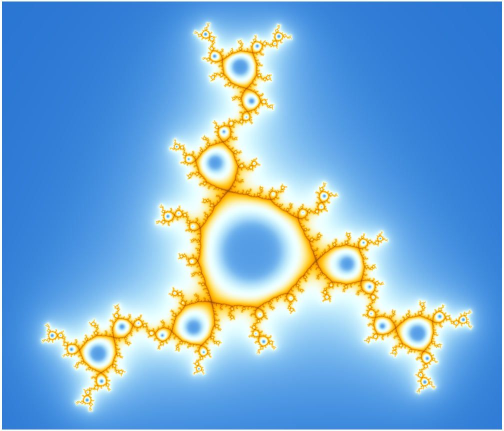
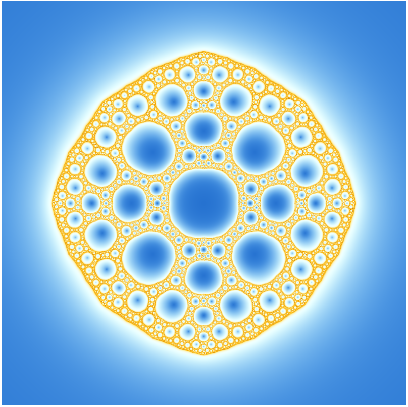


Julia set for $f(z) = z^3 - 0.21 + 1.09i$

Finitely Ramified Julia Sets

Julia sets for rational functions are sometimes finitely ramified.



Julia set for $f(z) = z^2 - \dfrac{1}{16z^2}$

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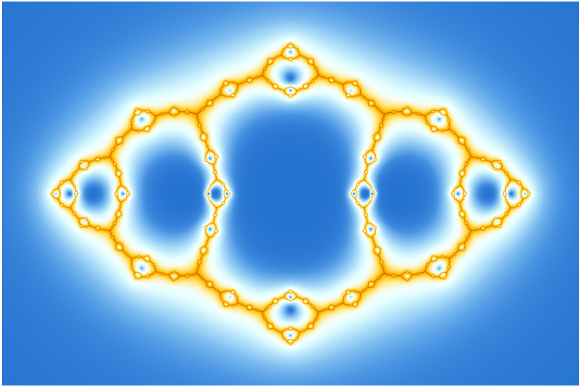

Julia set for $f(z) = \dfrac{1}{z^2} - 1$

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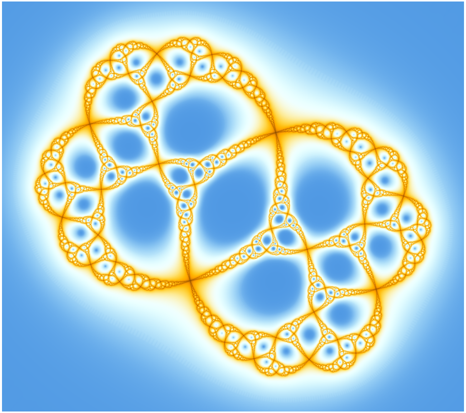

Julia set for $f(z) = \dfrac{e^{2\pi i/3}z^2 - 1}{z^2 - 1}$

Main Theorem

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A metric on a finitely ramified fractal X is ***undistorted*** if:

1. It has exponential cell decay, and
2. The cells have uniform relative separation.

Theorem (B–Forrest 2023)

1. *All undistorted metrics on X are quasisymmetrically equivalent.*
2. *Any metric quasisymmetrically equivalent to an undistorted metric is undistorted.*

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Exponential Cell Decay:

There exist constants $0 < r < R < 1$ and $C \geq 1$ so that

$$\frac{r^{|m-n|}}{C} \leq \frac{\operatorname{diam}(E')}{\operatorname{diam}(E)} \leq C R^{|m-n|}$$

for any m-cell E and n-cell E' that intersect.

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Uniform Relative Separation:
There exists a constant $\delta > 0$ so that

$$d(E, E') \geq \delta \operatorname{diam}(E)$$

for any two n-cells E and E' that are disjoint.

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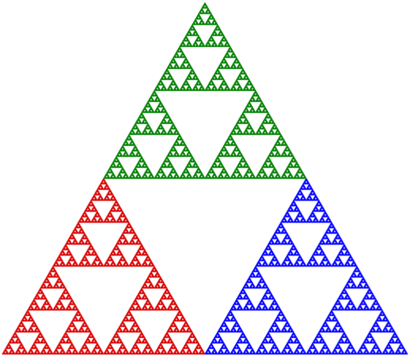
Corollary

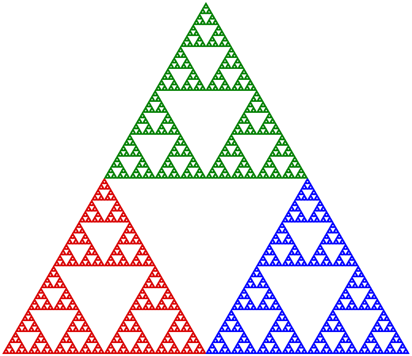
If X and Y have undistorted metrics, a homeomorphism $f : X \longrightarrow Y$ is a quasisymmetry if and only if the pushforward of the metric on X is undistorted.

Application: Sierpiński Triangles



Application: Sierpiński Triangles

Bandt and Retta (1992) proved that the Sierpiński triangle T is ***topologically rigid***, i.e. every homeomorphism of T maps n-cells to n-cells.



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Uniformization Theorem (B–Forrest 2023)

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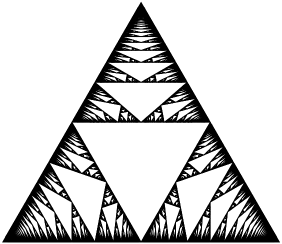

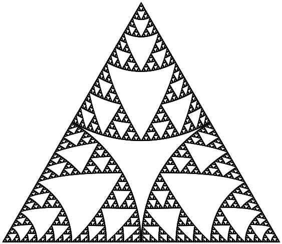

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We obtain a similar uniformization theorem for any topologically rigid fractal.

Applications to Julia Sets

Hyperbolic Functions

A rational function $f(z)$ is ***hyperbolic*** if the forward orbit of each critical point converges to an attracting cycle.

Such maps are expanding on their Julia set with respect to an appropriate metric.

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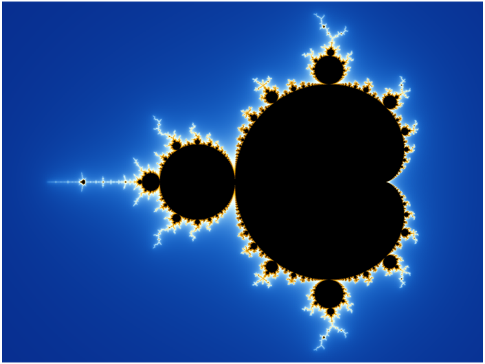

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All of our results apply only to hyperbolic rational functions f whose Julia sets J_f are connected.

Defining Cells

A set $S \subset J_f$ is a ***branch cut*** if f^{-1} has a single-valued branch on each component of $J_f \setminus S$.

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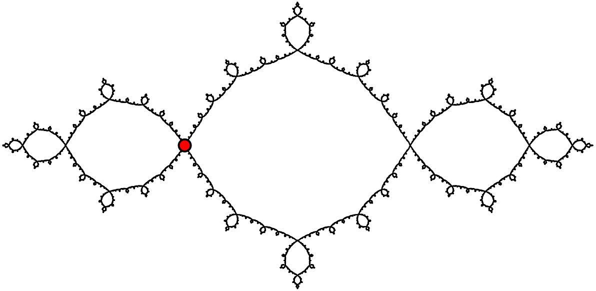

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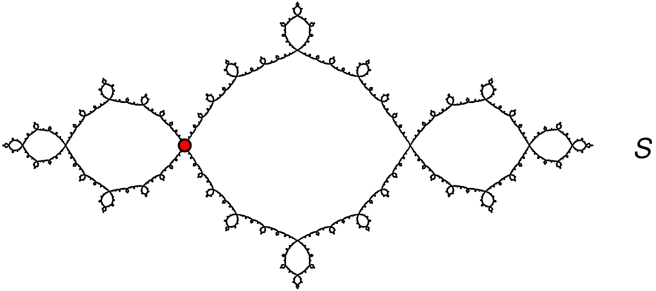
A set $S \subset J_f$ is a ***branch cut*** if f^{-1} has a single-valued branch on each component of $J_f \setminus S$.

If S is finite and invariant (i.e. $f(S) \subseteq S$), then the iterated preimages $f^{-n}(S)$ cut J_f into cells.



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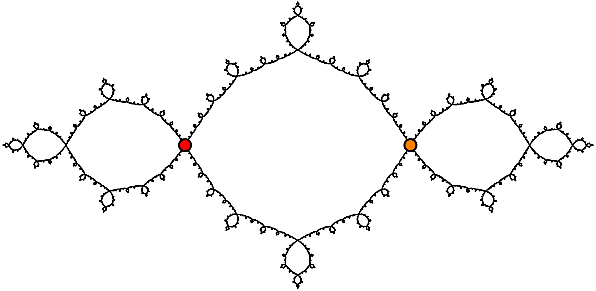

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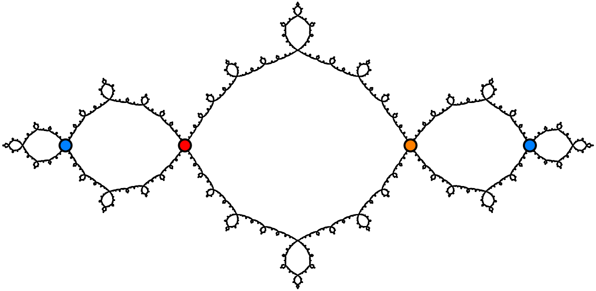

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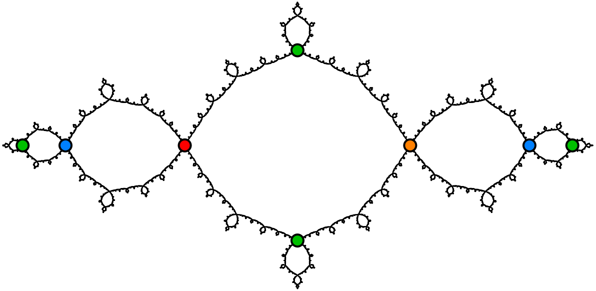

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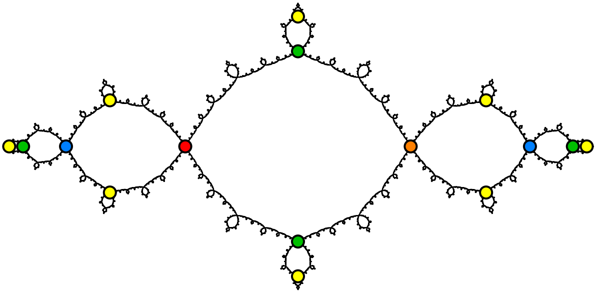

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If J_f has a finite invariant branch cut, the resulting cells define a finitely ramified cell structure on J_f, and the restriction of the spherical metric is undistorted.

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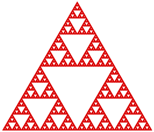
Note: In the polynomial case, a finite invariant branch cut always exists.

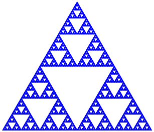
Constructing Quasisymmetries

Constructing Quasisymmetries

Consider two cells in a finitely ramified fractal X:



$\longrightarrow$



E (m-cell) E' (n-cell)

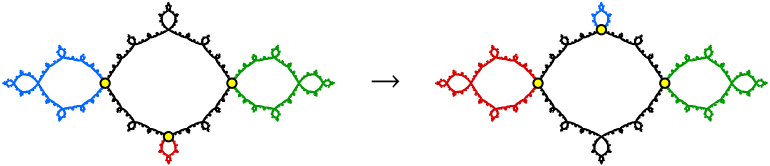
A homeomorphism $E \longrightarrow E'$ is ***cellular*** if it maps $(m+k)$-cells to $(n+k)$-cells for all $k \geq 0$.

Constructing Quasisymmetries

A homeomorphism of X is ***piecewise-cellular*** if there exist subdivisions

$$\{E_1, \ldots, E_n\} \qquad \text{and} \qquad \{E'_1, \ldots, E'_n\}$$

of X into cells so that each E_i maps to E'_i by a cellular map.



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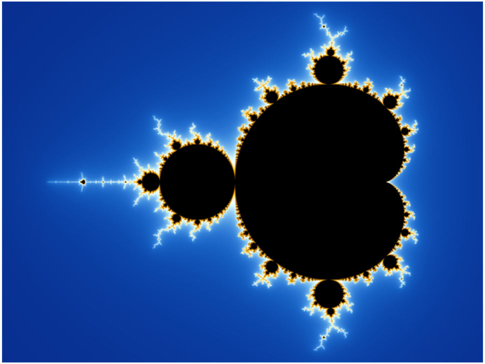
If the metric on X is undistorted, then any piecewise-cellular homeomorphism of X is a quasisymmetry.

This lets us construct quasisymmetries for many different Julia sets.

Main Results for Julia Sets

Theorem (B–Forrest 2023)

Any connected Julia set for a hyperbolic quadratic polynomial has infinitely many quasisymmetries.



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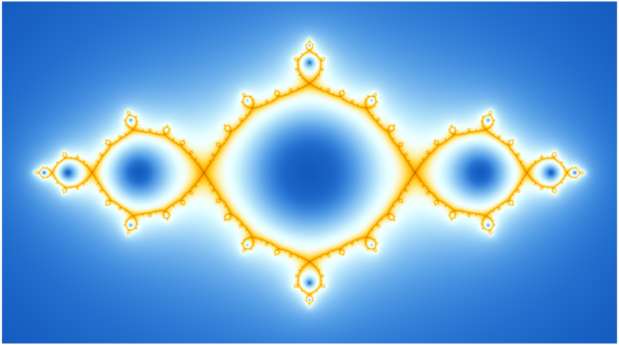

Julia set for $f(z) = z^2 - 1$

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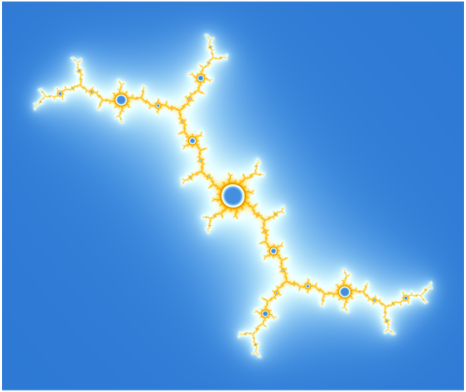

Julia set for $f(z) = z^2 - 0.157 + 1.032i$

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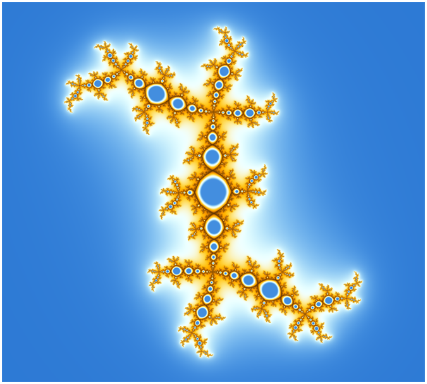

Julia set for $f(z) = z^2 + 0.32 + 0.56i$

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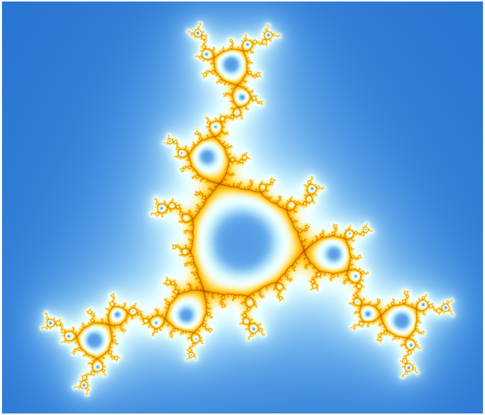
Specifically, we use “ping-pong lemmas” to show that the quasisymmetry group contains:

- A free product $\mathbb{Z}_2 * \mathbb{Z}_n$ for some $n \geq 2$, and
- Thompson’s group F.

All of our constructed quasisymmetries are piecewise-cellular.

Main Results for Julia Sets

We can also show that many other finitely ramified Julia sets have infinite quasisymmetry group.



Julia set for $f(z) = z^3 - 0.21 + 1.09i$

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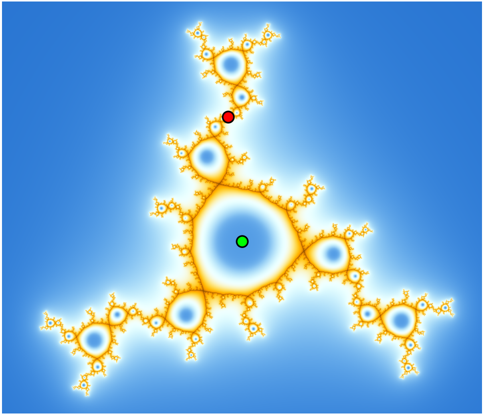

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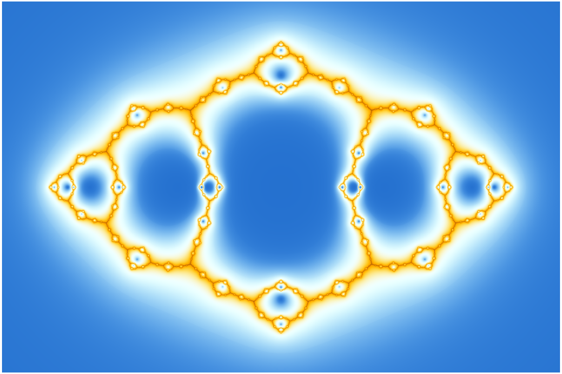
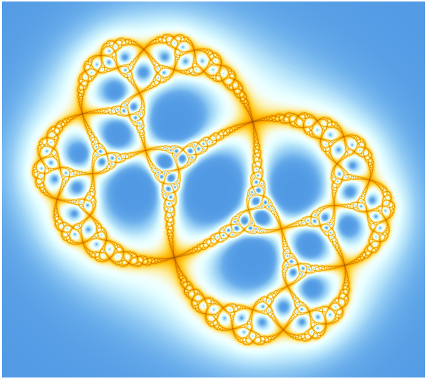


contains T

Julia set for $f(z) = \dfrac{1}{z^2} - 1$

Main Results for Julia Sets

However, some hyperbolic rational functions have a finitely ramified Julia set with only finitely many homeomorphisms.

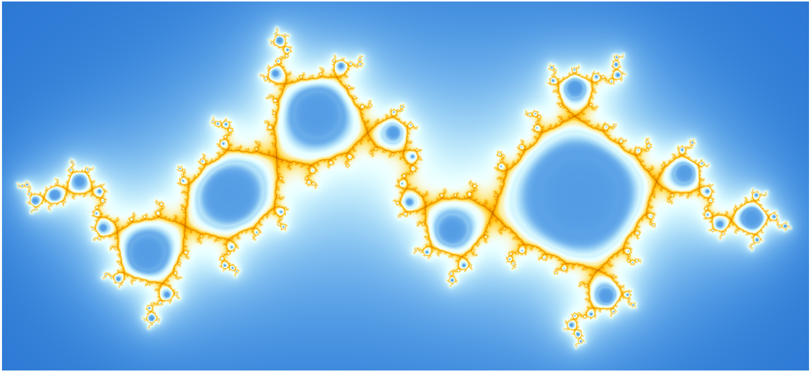


dihedral of order 8

Julia set for $f(z) = \dfrac{e^{2\pi i/3}z^2 - 1}{z^2 - 1}$

Main Results for Julia Sets

Also, we conjecture that some hyperbolic polynomials have Julia sets with finite quasisymmetry group.



Julia set for $f(z) = (4.424 + 1.374i)(z^3 - 3z + 2) - 1$

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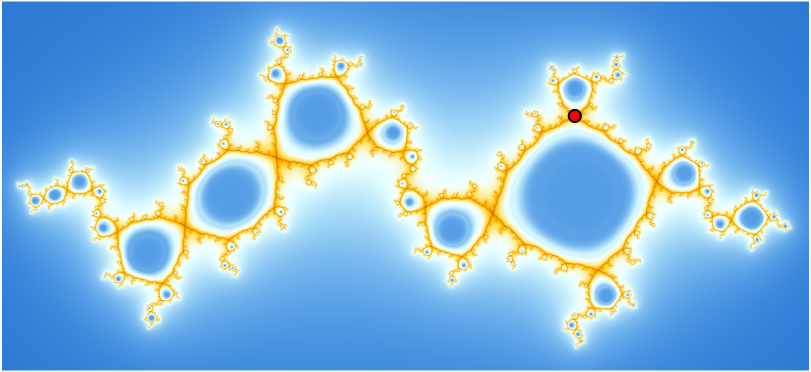

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The End